DETERMINING THE NORMALIZATION OF THE QUANTUM FIELD THEORY VACUUM, WITH IMPLICATIONS FOR QUANTUM GRAVITY

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In a standard quantum field theory the norm $\langle\Omega|\Omega\rangle$ of the vacuum state is taken, apparently without proof, to be finite. In this paper we provide a procedure, based on constructing an equivalent wave mechanics, for determining whether or not the vacuum norm is finite. We provide an example based on a second-order plus fourth-order scalar field theory, a prototype for quantum gravity, in which it is not. In this example the Minkowski path integral with a real measure diverges though the Euclidean path integral does not. Even though contributions from the Wick rotation contour are, again apparently without proof, ordinarily ignored, in this case they cannot be. Since $\langle\Omega|\Omega\rangle$ is not finite, use of the standard Feynman rules is not valid. And while these rules not only lead to states with negative norm, they in fact lead to states with infinite negative norm. However, if the fields in that theory are continued into the complex plane, we show that then there is a domain in the complex plane known as a Stokes wedge in which one can define an appropriate time-independent, positive and finite inner product, viz. the $\langle L|R\rangle$ overlap of left-eigenstates and right-eigenstates of the Hamiltonian; with the vacuum state then being normalizable, and with there being no states with negative or infinite $\langle L|R\rangle$ norm. In this Stokes wedge it is the Euclidean path integral that diverges while the Minkowski path integral does not. The concerns that we raise in this paper only apply to bosons since the matrices associated with their creation and annihilation operators are infinite dimensional. Since the ones associated with fermions are finite dimensional, the fermion theory vacuum is automatically normalizable. We discuss some general implications of our results for quantum gravity studies.

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Outline

1. Is $\langle\Omega|\Omega\rangle$ finite? How could it not be? Specifying an action and canonical commutators does not fix a Hilbert space. Possibly there might not be one. The path integral may not exist in any complex domain for the measure. Does PT symmetry guarantee that a domain does exist?

2. We present a procedure to determine whether or not $\langle\Omega|\Omega\rangle$ is finite, and show that it is for a standard second-order derivative bosonic field theory. The procedure enables us to write the quantum field theory Hamiltonian as a first-quantized derivative operator.

3. We show that $\langle\Omega|\Omega\rangle$ is not finite for a fourth-order derivative bosonic field theory.

4. We show that $\langle\Omega^L|\Omega^R\rangle = \langle\Omega^{PT}|\Omega\rangle$ is finite for a fourth-order derivative bosonic field theory.

5. We show that $\langle\Omega|\Omega\rangle$ is finite for fermion theory.

6. We discuss our results from the perspective of path integrals and the Wick rotation to the Euclidean case, and show that in the fourth-order derivative bosonic field theory case the contribution of the Wick rotation contour circle at infinity is not only not zero, it is infinite.

7. We discuss our results from the perspective of the Dyson-Wick expansion.

8. We discuss the implications of our results for constructing a consistent theory of quantum gravity.

1 The hidden assumption of quantum field theory

Consider a free relativistic neutral scalar field with action

$$I_{\rm S}=\int d^4x\tfrac{1}{2}\left[\partial_\mu\phi\partial^\mu\phi-m^2\phi^2\right], \tag{1.1}$$

and wave equation, Hamiltonian, and equal time commutation relation of the form

$$\begin{aligned}
[\partial_\mu\partial^\mu+m^2]\phi=0,\\
H=\int d^3x\tfrac{1}{2}[\dot{\phi}^2+\bar{\nabla}\phi\cdot\bar{\nabla}\phi+m^2\phi^2],\\
[\phi(\bar{x},t),\dot{\phi}(\bar{x}',t)]=i\delta^3(\bar{x}-\bar{x}').
\end{aligned} \tag{1.2}$$

With $\omega_k=+(\bar{k}^2+m^2)^{1/2}$ solutions to the wave equation obey

$$\phi(\bar{x},t)=\int\frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}}[a(\bar{k})e^{-i\omega_k t+i\bar{k}\cdot\bar{x}}+a^\dagger(\bar{k})e^{i\omega_k t-i\bar{k}\cdot\bar{x}}], \tag{1.3}$$

and with $[a(\bar{k}),a^\dagger(\bar{k}')]=\delta^3(\bar{k}-\bar{k}')$ the Hamiltonian is given by

$$H=\frac{1}{2}\int d^3k[\bar{k}^2+m^2]^{1/2}\left[a^\dagger(\bar{k})a(\bar{k})+a(\bar{k})a^\dagger(\bar{k})\right]. \tag{1.4}$$

Given (1.4) we can introduce a no-particle state $|\Omega\rangle$ that obeys $a(\bar{k})|\Omega\rangle=0$ for each $\bar{k}$, and can identify it as the ground state of H.

This procedure does not specify the value of $\langle\Omega|\Omega\rangle$.

For the theory the associated c-number propagator obeys

$$(\partial_t^2 - \bar{\nabla}^2 + m^2)D(x) = -\delta^4(x), \tag{1.5}$$

so that

$$D(x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot x}}{(k^2 - m^2 + i\epsilon)}. \tag{1.6}$$

If we identify the propagator as a vacuum matrix element of q-number fields, viz.

$$D(x) = -i\langle\Omega|T[\phi(x)\phi(0)]|\Omega\rangle, \tag{1.7}$$

then use of the equal commutation relation gives

$$(\partial_t^2 - \bar{\nabla}^2 + m^2)(-i)\langle\Omega|T[\phi(x)\phi(0)]|\Omega\rangle = -\langle\Omega|\Omega\rangle\delta^4(x). \tag{1.8}$$

Comparing with (1.5) we see that we can only identify $D(x)$ as the matrix element $-i\langle\Omega|T[\phi(x)\phi(0)]|\Omega\rangle$ if the vacuum is normalized to one, viz. $\langle\Omega|\Omega\rangle = 1$.

Now if the normalization of the vacuum is finite we of course can always rescale it to one. However, that presupposes that the normalization of the vacuum is not infinite. We are not aware of any proof in the literature that the normalization of the vacuum is not infinite (either in this particular case or in general), and taking it to be finite is a **hidden assumption**.

So we shall present a procedure for determining whether the normalization of the vacuum state is finite or infinite. **The procedure is based on generalizing to quantum field theory what we know from quantum mechanics.**

2 The quantum-mechanical simple harmonic oscillator

For a simple harmonic oscillator with Hamiltonian $H = \frac{1}{2}[p^2 + q^2]$ and commutator $[q, p] = i$, there are two sets of bases, the wave function basis and the occupation number space basis.

The wave function basis is obtained by setting $p = -i\partial/\partial q$ in H and then solving the Schrödinger wave equation $H\psi(q) = E\psi(q)$. In this way we obtain a ground state with energy $E_0 = \frac{1}{2}$ and wave function $\psi_0(q) = e^{-q^2/2}$.

For occupation number space we set $q = (a + a^\dagger)/\sqrt{2}$ and $p = i(a^\dagger - a)/\sqrt{2}$. This yields $[a, a^\dagger] = 1$ and $H = a^\dagger a + 1/2$. We introduce a no-particle state $|\Omega\rangle$ that obeys $a|\Omega\rangle = 0$, with $|\Omega\rangle$ being the occupation number space ground state with energy $E_0 = \frac{1}{2}$. However, in and of itself this does not fix the norm $\langle\Omega|\Omega\rangle$ of the no-particle state or oblige it to be finite.

To fix the $\langle\Omega|\Omega\rangle$ norm we need to relate the ground states of the two bases. With $a = (q + ip)/\sqrt{2}$ we set

$$\langle q|a|\Omega\rangle = \frac{1}{\sqrt{2}}\left(q + \frac{\partial}{\partial q}\right)\langle q|\Omega\rangle = 0, \tag{2.1}$$

and find that $\langle q|\Omega\rangle = e^{-q^2/2}$. We thus identify $\psi_0(q) = \langle q|\Omega\rangle$. We now calculate the standard Dirac norm for vacuum, and obtain

$$\langle\Omega|\Omega\rangle = \int_{-\infty}^{\infty} dq\langle\Omega|q\rangle\langle q|\Omega\rangle = \int_{-\infty}^{\infty} dq\psi_0^*(q)\psi_0(q) = \int_{-\infty}^{\infty} dq e^{-q^2} = \sqrt{\pi}. \tag{2.2}$$

We thus establish that the Dirac norm of the no-particle state is finite. And on setting $\psi_0(q) = e^{-q^2/2}/\pi^{1/4}$ we normalize it to one.

That we are able to do this is because we know the form of the wave function $\psi_0(q)$.

While this procedure is both straightforward and familiar, it works because both the wave function basis approach and occupation number basis approach have something in common:

they are both based on an infinite number of degrees of freedom.

For the occupation number basis we can represent the creation and annihilation operators as infinite-dimensional matrices labeled by $|\Omega\rangle$, $a^\dagger|\Omega\rangle$, $a^{\dagger 2}|\Omega\rangle$ and so on.

For the wave function basis the coordinate q is a continuous variable that varies between $-\infty$ and ∞.

The two sets of bases are both infinite dimensional, one discrete and the other continuous.

The advantage of the continuous basis is that it enables to us to express the normalization of the vacuum state as an integral with an infinite range, an integral that is then either finite or infinite.

For field theory we already have an occupation number space basis for the Hamiltonian. So can we write it as a wave operator?

3 The quantum field theory oscillator

In the quantum field theory case we do not know the form of the wave function solutions to $H|\psi\rangle = E|\psi\rangle$, since we cannot realize the canonical commutator $[\phi(\bar{x},t),\dot{\phi}(\bar{x}',t)] = i\delta^3(\bar{x}-\bar{x}')$ as a differential relation. Specifically, we cannot satisfy it by setting $\dot{\phi}(\bar{x},t)$ equal to $-i\partial/\partial\phi(\bar{x},t)$ (though we could introduce a functional derivative $\dot{\phi}(\bar{x},t) = -i\delta/\delta\phi(\bar{x},t)$).

However, we can express the Hamiltonian in terms of creation and annihilation operators. So what we can then do is **reverse engineer** what we did in the quantum-mechanical case. For each $\bar{k}$ we thus introduce

$$a(\bar{k}) = \frac{1}{\sqrt{2}}[q(\bar{k}) + ip(\bar{k})], \quad a^\dagger(\bar{k}) = \frac{1}{\sqrt{2}}[q(\bar{k}) - ip(\bar{k})], \tag{3.1}$$

so that

$$[q(\bar{k}),p(\bar{k}')] = i\delta^3(\bar{k}-\bar{k}'), \quad H = \frac{1}{2}\int d^3k[\bar{k}^2+m^2]^{1/2}[p^2(\bar{k})+q^2(\bar{k})],$$
$$\phi(\bar{x},t) = \frac{1}{\sqrt{2}}\int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}}\left[[q(\bar{k})+ip(\bar{k})]e^{-i\omega_k t+i\bar{k}\cdot\bar{x}} + [q(\bar{k})-ip(\bar{k})]e^{i\omega_k t-i\bar{k}\cdot\bar{x}}\right]. \tag{3.2}$$

These $q(\bar{k})$ and $p(\bar{k})$ operators bear no relation to any physical position or momentum operators. Their only role here is to enable us to convert the discrete infinite-dimensional basis associated with each $a(\bar{k})$ and $a^\dagger(\bar{k})$ into a continuous one. Specifically, we can realize the $[q(\bar{k}),p(\bar{k}')]$ commutator by $p(\bar{k}') = -i\partial/\partial q(\bar{k}')$, with H then becoming a wave operator. In this way for each $\bar{k}$ we obtain a solution to the Schrödinger equation of the form $\psi(\bar{k}) = e^{-q^2(\bar{k})/2}/\pi^{1/4}$. We can define a no-particle vacuum that obeys $a(\bar{k})|\Omega\rangle$ for each $\bar{k}$. For each $\bar{k}$ we have

$$\langle q(\bar{k})|a(\bar{k})|\Omega\rangle = \frac{1}{\sqrt{2}}\left[q(\bar{k}) + \frac{\partial}{\partial q(\bar{k})}\right]\langle q(\bar{k})|\Omega\rangle = 0, \tag{3.3}$$

so that $\langle q(\bar{k})|\Omega\rangle = e^{-q^2(\bar{k})/2}/\pi^{1/4}$, and thus

$$\langle\Omega|\Omega\rangle = \Pi_{\bar{k}}\int dq(\bar{k})\langle\Omega|q(\bar{k})\rangle\langle q(\bar{k})|\Omega\rangle = \Pi_{\bar{k}}\int dq(\bar{k})\frac{e^{-q^2(\bar{k})}}{\pi^{1/2}} = \Pi_{\bar{k}}1 = 1. \tag{3.4}$$

Thus the vacuum for the full H obeys $\langle\Omega|\Omega\rangle = 1$, to thus have a finite normalization. In this way we establish that the vacuum state of the free relativistic scalar field is normalizable.

The general prescription then is to convert the occupation number space Hamiltonian into a product of individual occupation number spaces each with its own $\bar{k}$, and then determine whether the equivalent wave mechanics ground state wave functions constructed this way have a finite normalization in the conventional Schrödinger wave mechanics theory sense. If they do, then so does the full vacuum $|\Omega\rangle$ of the full H. If on the other hand the equivalent wave mechanics wave functions are not normalizable, then neither is the full $|\Omega\rangle$.

Once we are able to show that the vacuum state of the free theory is normalizable, this will remain true in the presence of interactions if the interacting theory is renormalizable. Specifically, if the free theory $D(x) = -i\langle\Omega|T[\phi(x)\phi(0)]|\Omega\rangle$ is finite, which it will be if the free theory $\langle\Omega|\Omega\rangle$ is, then the interacting $D(x) = -i\langle\Omega|T[\phi(x)\phi(0)]|\Omega\rangle$ propagator will equally be finite after renormalization. Consequently, the renormalized $\langle\Omega|\Omega\rangle$ will be finite too. Thus to establish the finiteness of the vacuum normalization of a renormalizable interacting theory, we only need to be able to make a creation and annihilation representation of the free theory. We discuss the role of interactions further below.

Since the above analysis is driven by the fact that the dimension of the occupation number space is infinite, the analysis can be carried out for any bosonic field. However, because of the Pauli principle, the occupation number space basis for fermions of any given $\bar{k}$ is finite dimensional. Thus we have to treat fermions separately, and do so below.

Having found a theory for which $\langle\Omega|\Omega\rangle$ is finite, we present an example for which $\langle\Omega|\Omega\rangle$ is not finite. The example is based on a second-order plus fourth-order derivative neutral scalar field theory. As we discuss below, this model is associated with radiative corrections in quantum gravity.

4 Higher-derivative quantum field theories

The action and equation of motion are of the form

$$I_S = \frac{1}{2}\int d^4x\Big[\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi-(M_1^2+M_2^2)\partial_\mu\phi\partial^\mu\phi+M_1^2M_2^2\phi^2\Big],$$
$$(\partial_t^2-\bar{\nabla}^2+M_1^2)(\partial_t^2-\bar{\nabla}^2+M_2^2)\phi(x)=0. \tag{4.1}$$

The associated propagator obeys the ghost-like

$$(\partial_t^2-\bar{\nabla}^2+M_1^2)(\partial_t^2-\bar{\nabla}^2+M_2^2)D(x)=-\delta^4(x),$$
$$D(x)=-\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot x}}{(k^2-M_1^2)(k^2-M_2^2)}=-\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot x}}{(M_1^2-M_2^2)}\left[\frac{1}{(k^2-M_1^2)}-\frac{1}{(k^2-M_2^2)}\right]. \tag{4.2}$$

The energy-momentum tensor $T_{\mu\nu}$, the canonical momenta π^μ and $\pi^{\mu\lambda}$, and the equal-time commutators appropriate to the higher-derivative theory are given by (Bender and Mannheim 2008)

$$T_{\mu\nu}=\pi_\mu\phi_{,\nu}+\pi_\mu{}^\lambda\phi_{,\nu,\lambda}-\eta_{\mu\nu}\mathcal{L},$$
$$\pi^\mu=\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}-\partial_\lambda\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\mu,\lambda}}\right)=-\partial_\lambda\partial^\mu\partial^\lambda\phi-(M_1^2+M_2^2)\partial^\mu\phi,$$
$$\pi^{\mu\lambda}=\frac{\partial\mathcal{L}}{\partial\phi_{,\mu,\lambda}}=\partial^\mu\partial^\lambda\phi,$$
$$T_{00}=\tfrac{1}{2}\pi_{00}^2+\pi_0\dot{\phi}+\tfrac{1}{2}(M_1^2+M_2^2)\dot{\phi}^2-\tfrac{1}{2}M_1^2M_2^2\phi^2-\tfrac{1}{2}\pi_{ij}\pi^{ij}+\tfrac{1}{2}(M_1^2+M_2^2)\phi_{,i}\phi^{,i}$$
$$=\frac{1}{2}\ddot{\phi}^2-\tfrac{1}{2}(M_1^2+M_2^2)\dot{\phi}^2-\dddot{\phi}\dot{\phi}-[\partial_i\partial^i\dot{\phi}]\dot{\phi}-\tfrac{1}{2}M_1^2M_2^2\phi^2-\tfrac{1}{2}\partial_i\partial_j\phi\partial^i\partial^j\phi+\tfrac{1}{2}(M_1^2+M_2^2)\partial_i\phi\partial^i\phi,$$
$$[\phi(\bar{0},t),\dot{\phi}(\bar{x},t)]=0,\qquad[\phi(\bar{0},t),\ddot{\phi}(\bar{x},t)]=0,\qquad[\phi(\bar{0},t),\dddot{\phi}(\bar{x},t])=-i\delta^3(x). \tag{4.3}$$

With the use of these commutation relations we find that

$$D(x)=i\langle\Omega|T[\phi(x)\phi(0)]|\Omega\rangle \tag{4.4}$$

indeed satisfies the first equation given in (4.2), **provided that is that $\langle\Omega|\Omega\rangle=1$. And if $\langle\Omega|\Omega\rangle=\infty$, then Wick's theorem and the associated Feynman rules are not valid.**

To check whether $\langle\Omega|\Omega\rangle$ actually is finite, we need to express the scalar field Hamiltonian $H_S = \int d^3x T_{00}$ in terms of creation and annihilation operators and then construct an equivalent wave mechanics. Given that the solutions to to the wave equation are plane waves, we set

$$\phi(\bar{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}} \left[a_1(\bar{k})e^{-i\omega_1 t+i\bar{k}\cdot\bar{x}} + a_1^\dagger(\bar{k})e^{i\omega_1 t-i\bar{k}\cdot\bar{x}} + a_2(\bar{k})e^{-i\omega_2 t+i\bar{k}\cdot\bar{x}} + a_2^\dagger(\bar{k})e^{i\omega_2 t-i\bar{k}\cdot\bar{x}}\right]. \tag{4.5}$$

where $\omega_1 = +(\bar{k}^2 + M_1^2)^{1/2}$, $\omega_2 = +(\bar{k}^2 + M_2^2)^{1/2}$. Given the commutators in (4.3) we obtain

$$\begin{aligned}
&[a_1(\bar{k}), a_1^\dagger(\bar{k}')] = [2(M_1^2 - M_2^2)(\bar{k}^2 + M_1^2)^{1/2}]^{-1}\delta^3(\bar{k} - \bar{k}'),\\
&[a_2(\bar{k}), a_2^\dagger(\bar{k}')] = -[2(M_1^2 - M_2^2)(\bar{k}^2 + M_2^2)^{1/2}]^{-1}\delta^3(\bar{k} - \bar{k}'),\\
&[a_1(\bar{k}), a_2(\bar{k}')] = 0, \quad [a_1(\bar{k}), a_2^\dagger(\bar{k}')] = 0, \quad [a_1^\dagger(\bar{k}), a_2(\bar{k}')] = 0, \quad [a_1^\dagger(\bar{k}), a_2^\dagger(\bar{k}')] = 0,
\end{aligned} \tag{4.6}$$

with the Hamiltonian then taking the form

$$\begin{aligned}
H_S &= \frac{1}{2}\int d^3k \Big[2(M_1^2 - M_2^2)(\bar{k}^2 + M_1^2)\left[a_1^\dagger(\bar{k})a_1(\bar{k}) + a_1(\bar{k})a_1^\dagger(\bar{k})\right]\\
&- 2(M_1^2 - M_2^2)(\bar{k}^2 + M_2^2)\left[a_2^\dagger(\bar{k})a_2(\bar{k}) + a_2(\bar{k})a_2^\dagger(\bar{k})\right]\Big]\\
&= \int d^3k \Big[2(M_1^2 - M_2^2)(\bar{k}^2 + M_1^2)a_1^\dagger(\bar{k})a_1(\bar{k}) - 2(M_1^2 - M_2^2)(\bar{k}^2 + M_2^2)a_2^\dagger(\bar{k})a_2(\bar{k})\\
&+ \frac{1}{2}(\bar{k}^2 + M_1^2)^{1/2}\delta^3(0) + \frac{1}{2}(\bar{k}^2 + M_2^2)^{1/2}\delta^3(0)\Big],
\end{aligned} \tag{4.7}$$

where $(2\pi)^3\delta^3(0)$ is a quantization box volume V. We note that with $M_1^2 - M_2^2 > 0$ for definitiveness, we see negative signs in both H_S and the $[a_2(\bar{k}), a_2^\dagger(\bar{k}')]$ commutator, while noting that despite this the zero-point energy is positive. We shall see below that the negative sign concerns will be resolved once we settle the issue of the normalization of the vacuum. To do that we now descend to the quantum-mechanical limit of the theory, the Pais-Uhlenbeck oscillator model.

5 Higher-derivative quantum mechanics

In order to study the Pauli-Villars regulator, in 1950 Pais and Uhlenbeck (PU) introduced a fourth-order quantum-mechanical oscillator model with action and equation of motion

$$I_{\rm PU} = \frac{1}{2}\int dt\left[\ddot{z}^2 - \left(\omega_1^2+\omega_2^2\right)\dot{z}^2 + \omega_1^2\omega_2^2 z^2\right], \qquad \ddddot{z} + (\omega_1^2+\omega_2^2)\ddot{z} + \omega_1\omega_2 z^2 = 0, \tag{5.1}$$

where for definitiveness in the following we take $\omega_1 > \omega_2$. As constructed this action possesses three variables z, $\dot{z}$ and $\ddot{z}$. This is too many for one oscillator but not enough for two. The system is thus a constrained system. And so we introduce a new variable $x = \dot{z}$ and its conjugate p_x. And using the method of Dirac constraints obtain the time-independent Hamiltonian (Mannheim and Davidson 2000, 2005)

$$H_{\rm PU} = \frac{p_x^2(t)}{2} + p_z(t)x(t) + \frac{1}{2}\left(\omega_1^2+\omega_2^2\right)x^2(t) - \frac{1}{2}\omega_1^2\omega_2^2 z^2(t), \tag{5.2}$$

with two sets of canonical equal-time commutators of the form

$$[z(t), p_z(t)] = i, \qquad [x(t), p_x(t)] = i. \tag{5.3}$$

The terms in $H_{\rm PU}$ are in complete parallel to the first four terms in the field theory T_{00}, viz. $\frac{1}{2}\pi_{00}^2 + \pi_0\dot{\phi} + \frac{1}{2}(M_1^2+M_2^2)\dot{\phi}^2 - \frac{1}{2}M_1^2M_2^2\phi^2$, with the PU oscillator model being the nonrelativistic limit of the relativistic scalar field theory, with the spatial dependence having been frozen out. Since canonical commutators only involve time derivatives, freezing out the spatial dependence will still give the full dynamical content of the relativistic theory. In fact we can set $i = [z, p_z] \equiv [\phi, \pi_0] = [\phi, -\dddot{\phi} - (M_1^2+M_2^2)\dot{\phi}] = i\delta^3(x)$, to thus parallel the $[\phi(\bar{0},t), \dot{\phi}(\bar{x},t)] = 0$, $[\phi(\bar{0},t), \ddot{\phi}(\bar{x},t)] = 0$, $[\phi(\bar{0},t), \dddot{\phi}(\bar{x},t]) = -i\delta^3(x)$ commutators.

On setting $p_z = -i\partial_z$, $p_x = -i\partial_x$ the Schrödinger problem for $H_{\rm PU}$ can be solved analytically, with the state with energy $(\omega_1+\omega_2)/2$ having a wave function that is of the form (Mannheim 2007)

$$\psi_0(z,x) = \exp[\tfrac{1}{2}(\omega_1+\omega_2)\omega_1\omega_2 z^2 + i\omega_1\omega_2 zx - \tfrac{1}{2}(\omega_1+\omega_2)x^2]. \tag{5.4}$$

While this wave function is well behaved at large x, it diverges at large z, and consequently as a wave function it is not normalizable.

To relate this wave function to the no-particle vacuum $|\Omega\rangle$ we second quantize the theory. And with the wave equation given in (5.1), and with $\dot{z}=i[H_{\rm PU},z]=x$, $\dot{x}=p_x$, $\dot{p}_x=-p_z-(\omega_1^2+\omega_2^2)x$, $\dot{p}_z=\omega_1^2\omega_2^2 z$, we obtain

$$
\begin{aligned}
z(t) &= a_1e^{-i\omega_1t}+a_1^{\dagger}e^{i\omega_1t}+a_2e^{-i\omega_2t}+a_2^{\dagger}e^{i\omega_2t},\\
p_z(t) &= i\omega_1\omega_2^2[a_1e^{-i\omega_1t}-a_1^{\dagger}e^{i\omega_1t}]+i\omega_1^2\omega_2[a_2e^{-i\omega_2t}-a_2^{\dagger}e^{i\omega_2t}],\\
x(t) &= -i\omega_1[a_1e^{-i\omega_1t}-a_1^{\dagger}e^{i\omega_1t}]-i\omega_2[a_2e^{-i\omega_2t}-a_2^{\dagger}e^{i\omega_2t}],\\
p_x(t) &= -\omega_1^2[a_1e^{-i\omega_1t}+a_1^{\dagger}e^{i\omega_1t}]-\omega_2^2[a_2e^{-i\omega_2t}+a_2^{\dagger}e^{i\omega_2t}],
\end{aligned}
\tag{5.5}
$$

and a Hamiltonian and commutator algebra of the form (Mannheim and Davidson 2000, 2005)

$$
H_{\rm PU}=2(\omega_1^2-\omega_2^2)(\omega_1^2a_1^{\dagger}a_1-\omega_2^2a_2^{\dagger}a_2)+\tfrac{1}{2}(\omega_1+\omega_2),
\tag{5.6}
$$

$$
[a_1,a_1^{\dagger}]=\frac{1}{2\omega_1(\omega_1^2-\omega_2^2)},\qquad [a_2,a_2^{\dagger}]=-\frac{1}{2\omega_2(\omega_1^2-\omega_2^2)}.
\tag{5.7}
$$

We note the similarity to (4.7) and (4.6).

Even though the $a_2^{\dagger}a_2$ term appears in $H_{\rm PU}$ with a minus sign, there is a compensating minus sign in the $[a_2,a_2^{\dagger}]$ commutator. In consequence, all energy eigenvalues of $H_{\rm PU}$ are positive, with the no-particle state $|\Omega\rangle$ that both a_1 and a_2 annihilate being the state of lowest energy. And with its energy being $(\omega_1+\omega_2)/2$, we can associate it with $\psi_0(z,x)e^{-i(\omega_1+\omega_2)t/2}$, with the normalization of $|\Omega\rangle$ then being given by

$$
\langle\Omega|\Omega\rangle=\int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}dx\langle\Omega|z,x\rangle\langle z,x|\Omega\rangle=\int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}dx\psi_0^*(z,x)\psi_0(z,x)=\infty.
\tag{5.8}
$$

With $\psi_0(z,x)$ diverging at large z, this normalization integral is infinite. Thus we see that through our knowledge of the form of the ground state wave function as given in (5.4) we are able to determine the normalization of the PU theory vacuum and establish that it is infinite. We can thus anticipate and will immediately show that this is also the case for the second-order plus fourth-order scalar quantum field theory as well. Then we will discuss what to do about it, with there actually being a mechanism for obtaining a finite normalization (Bender and Mannheim 2008), one that also takes care of the fact that according to (5.7) $\langle\Omega|a_2a_2^{\dagger}|\Omega\rangle$ is negative.

6 The nonnormalizable vacuum of higher-derivative field theories

To determine the second-order plus fourth-order scalar field theory vacuum normalization we first need to invert (5.5). This yields

$$
\begin{aligned}
a_1e^{-i\omega_1 t} &= \frac{1}{2(\omega_1^2-\omega_2^2)}\left[-\omega_2^2 z(t)-p_x(t)+i\omega_1 x(t)+i\frac{p_z(t)}{\omega_1}\right],\\
a_1^{\dagger}e^{i\omega_1 t} &= \frac{1}{2(\omega_1^2-\omega_2^2)}\left[-\omega_2^2 z(t)-p_x(t)-i\omega_1 x(t)-i\frac{p_z(t)}{\omega_1}\right],\\
a_2e^{-i\omega_2 t} &= \frac{1}{2(\omega_1^2-\omega_2^2)}\left[\omega_1^2 z(t)+p_x(t)-i\omega_2 x(t)-i\frac{p_z(t)}{\omega_2}\right],\\
a_2^{\dagger}e^{i\omega_2 t} &= \frac{1}{2(\omega_1^2-\omega_2^2)}\left[\omega_1^2 z(t)+p_x(t)+i\omega_2 x(t)+i\frac{p_z(t)}{\omega_2}\right].
\end{aligned}
\tag{6.1}
$$

On generalizing to each $\bar{k}$ and setting $\omega_1(\bar{k})=+(\bar{k}^2+M_1^2)^{1/2}$, $\omega_2(\bar{k})=+(\bar{k}^2+M_2^2)^{1/2}$, we obtain

$$
\begin{aligned}
a_1(\bar{k})e^{-i\omega_1(\bar{k})t} &= \frac{1}{2(M_1^2-M_2^2)}\left[-\omega_2^2(\bar{k})z(\bar{k},t)-p_x(\bar{k},t)+i\omega_1(\bar{k})x(\bar{k},t)+i\frac{p_z(\bar{k},t)}{\omega_1(\bar{k})}\right],\\
a_1^{\dagger}(\bar{k})e^{i\omega_1(\bar{k})t} &= \frac{1}{2(M_1^2-M_2^2)}\left[-\omega_2^2(\bar{k})z(\bar{k},t)-p_x(\bar{k},t)-i\omega_1(\bar{k})x(\bar{k},t)-i\frac{p_z(\bar{k},t)}{\omega_1(\bar{k})}\right],\\
a_2(\bar{k})e^{-i\omega_2(\bar{k})t} &= \frac{1}{2(M_1^2-M_2^2)}\left[\omega_1^2(\bar{k})z(\bar{k},t)+p_x(\bar{k},t)-i\omega_2(\bar{k})x(\bar{k},t)-i\frac{p_z(\bar{k},t)}{\omega_2(\bar{k})}\right],\\
a_2^{\dagger}(\bar{k})e^{i\omega_2(\bar{k})t} &= \frac{1}{2(M_1^2-M_2^2)}\left[\omega_1^2(\bar{k})z(\bar{k},t)+p_x(\bar{k},t)+i\omega_2(\bar{k})x(\bar{k},t)+i\frac{p_z(\bar{k},t)}{\omega_2(\bar{k})}\right].
\end{aligned}
\tag{6.2}
$$

Inverting (6.2) gives

$$
\begin{aligned}
z(\bar{k},t) &= a_1(\bar{k})e^{-i\omega_1(\bar{k})t}+a_1^{\dagger}(\bar{k})e^{i\omega_1(\bar{k})t}+a_2(\bar{k})e^{-i\omega_2(\bar{k})t}+a_2^{\dagger}(\bar{k})^{i\omega_2(\bar{k})t},\\
p_z(\bar{k},t) &= i\omega_1(\bar{k})\omega_2^2(\bar{k})[a_1(\bar{k})e^{-i\omega_1(\bar{k})t}-a_1^{\dagger}(\bar{k})e^{i\omega_1(\bar{k})t}]+i\omega_1^2(\bar{k})\omega_2(\bar{k})[a_2(\bar{k})e^{-i\omega_2(\bar{k})t}-a_2^{\dagger}(\bar{k})e^{i\omega_2(\bar{k})t}],\\
x(\bar{k},t) &= -i\omega_1(\bar{k})[a_1(\bar{k})e^{-i\omega_1(\bar{k})t}-a_1^{\dagger}(\bar{k})e^{i\omega_1(\bar{k})t}]-i\omega_2(\bar{k})[a_2(\bar{k})e^{-i\omega_2(\bar{k})t}-a_2^{\dagger}(\bar{k})^{i\omega_2(\bar{k})t}],\\
p_x(\bar{k},t) &= -\omega_1^2(\bar{k})[a_1(\bar{k})e^{-i\omega_1(\bar{k})t}+a_1^{\dagger}(\bar{k})e^{i\omega_1(\bar{k})t}]-\omega_2^2(\bar{k})[a_2(\bar{k})e^{-i\omega_2(\bar{k})t}+a_2^{\dagger}(\bar{k})^{i\omega_2(\bar{k})t}].
\end{aligned}
\tag{6.3}
$$

From (6.3) and the commutation relations given in (4.6) it follows that

$$
\begin{aligned}
&[z(\bar{k},t),p_z(\bar{k}',t)]=\delta^3(\bar{k}-\bar{k}'), \qquad [x(\bar{k},t),p_x(\bar{k}',t)]=\delta^3(\bar{k}-\bar{k}'),\\
&[z(\bar{k},t),x(\bar{k}',t)]=0, \quad [z(\bar{k},t),p_x(\bar{k}',t)]=0, \quad [p_z(\bar{k},t),x(\bar{k}',t)]=0, \quad [p_z(\bar{k},t),p_x(\bar{k}',t)]=0.
\end{aligned} \tag{6.4}
$$

Insertion of (6.2) into the Hamiltonian given in (4.7) then yields an equivalent, time-independent Hamiltonian

$$
H_S = \int d^3k\left[\frac{p_x^2(\bar{k},t)}{2}+p_z(\bar{k},t)x(\bar{k},t)+\frac{1}{2}\left[\omega_1^2(\bar{k})+\omega_2^2(\bar{k})\right]x^2(\bar{k},t)-\frac{1}{2}\omega_1^2(\bar{k})\omega_2^2(\bar{k})z^2(\bar{k},t)\right]. \tag{6.5}
$$

For each momentum state we recognize the quantum field theory Hamiltonian H_S given in (6.5) as being of precisely the form of the quantum-mechanical $H_{\rm PU}$ Hamiltonian that is given in (5.2).

We can now proceed as in the second-order case and represent the commutators by

$$
\left[z(\bar{k},t),-i\frac{\partial}{\partial z(\bar{k}',t)}\right]=\delta^3(\bar{k}-\bar{k}'), \qquad \left[x(\bar{k},t),-i\frac{\partial}{\partial x(\bar{k}',t)}\right]=\delta^3(\bar{k}-\bar{k}'). \tag{6.6}
$$

With the vacuum obeying $a_1(\bar{k})|\Omega\rangle=0$, $a_2(\bar{k})|\Omega\rangle=0$ for each $\bar{k}$, from (6.2) we obtain

$$
\begin{aligned}
\langle z(\bar{k}),x(\bar{k})|a_1(\bar{k})|\Omega\rangle &= \frac{1}{2(M_1^2-M_2^2)}\left[-\omega_2^2(\bar{k})z(\bar{k})+i\frac{\partial}{\partial x(\bar{k})}+i\omega_1(\bar{k})x(\bar{k})+\frac{1}{\omega_1(\bar{k})}\frac{\partial}{\partial z(\bar{k})}\right]\langle z(\bar{k}),x(\bar{k})|\Omega\rangle=0,\\
\langle z(\bar{k}),x(\bar{k})|a_2(\bar{k})|\Omega\rangle &= \frac{1}{2(M_1^2-M_2^2)}\left[\omega_1^2(\bar{k})z(\bar{k})-i\frac{\partial}{\partial x(\bar{k})}-i\omega_2(\bar{k})x(\bar{k})-\frac{1}{\omega_2(\bar{k})}\frac{\partial}{\partial z(\bar{k})}\right]\langle z(\bar{k}),x(\bar{k})|\Omega\rangle=0,
\end{aligned} \tag{6.7}
$$

for each $\bar{k}$. From (6.7) it follows that for each $\bar{k}$ we can identify each $\langle z(\bar{k}),x(\bar{k})|\Omega\rangle$ with the PU oscillator ground state wave function $\psi_0(z(\bar{k}),x(\bar{k}))$, which, analogously to (5.4), is given by

$$
\psi_0(z(\bar{k}),x(\bar{k}))=\exp[\tfrac{1}{2}[\omega_1(\bar{k})+\omega_2(\bar{k})]\omega_1(\bar{k})\omega_2(\bar{k})z^2(\bar{k})+i\omega_1(\bar{k})\omega_2(\bar{k})z(\bar{k})x(\bar{k})-\tfrac{1}{2}[\omega_1(\bar{k})+\omega_2(\bar{k})]x^2(\bar{k})]. \tag{6.8}
$$

Consequently, the normalization of the vacuum is given by

$$
\begin{aligned}
\langle\Omega|\Omega\rangle &= \Pi_{\bar{k}}\int_{-\infty}^{\infty}dz(\bar{k})\int_{-\infty}^{\infty}dx(\bar{k})\langle\Omega|z(\bar{k}),x(\bar{k})\rangle\langle z(\bar{k}),x(\bar{k})|\Omega\rangle\\
&= \Pi_{\bar{k}}\int_{-\infty}^{\infty}dz(\bar{k})\int_{-\infty}^{\infty}dx(\bar{k})\psi_0^*(z(\bar{k}),x(\bar{k}))\psi_0(z(\bar{k}),x(\bar{k})).
\end{aligned} \tag{6.9}
$$

With each $\psi_0(z(\bar{k}),x(\bar{k}))$ diverging at large $z(\bar{k})$, **we thus establish that the normalization of the field theory vacuum is infinite**. Thus whatever is the normalization of the vacuum in the associated wave-mechanical limit translates into the same normalization in the quantum field theory.

7 How to obtain a normalizable vacuum

In analyzing the second-order plus fourth-order scalar field theory we note that with a conventional Hermitian field $\phi(x)$, and thus with $a_1^\dagger(\bar{k})$ and $a_2^\dagger(\bar{k})$ being the Hermitian conjugates of $a_1(\bar{k})$ and $a_2(\bar{k})$, the $a_2^\dagger(\bar{k})a_2(\bar{k})$ product would be positive definite and the energy spectrum of H_S as given in (4.7) would initially be unbounded from below, this being the familiar Ostrogradski instability of higher-derivative theories with Hermitian fields.

However, from (4.6) we see that $\langle\Omega|a_2(\bar{k})a_2^\dagger(\bar{k})|\Omega\rangle$ would be negative. This would imply the presence of ghost states of negative norm, with it then not being the case that a product such as $a_2(\bar{k})a_2^\dagger(\bar{k})$ could be positive definite. If one accepts this then matrix elements of the $-2(M_1^2-M_2^2)(\bar{k}^2+M_2^2)a_2^\dagger(\bar{k})a_2(\bar{k})$ term in H_S would be compensated for by the ghost signature, and the energy spectrum of H_S would then be bounded from below.

While this takes care of the unboundedness of the energy spectrum, it does so at a high price, namely the presence of unitarity-violating ghost states. But if $a_2^\dagger(\bar{k})$ is the Hermitian conjugate of $a_2(\bar{k})$ then $\langle\Omega|a_2(\bar{k})a_2^\dagger(\bar{k})|\Omega\rangle$ would have to be positive. Thus despite the dagger notation $a_2^\dagger(\bar{k})$ could not be the Hermitian conjugate of $a_2(\bar{k})$.

Hence our starting assumption that $\phi(x)$ is Hermitian could not be valid. Consequently, the Hamiltonian that is built out of the $\phi(x)$ field could not be Hermitian either. And in fact we have actually established that it is not, since the diverging of $\psi_0(z(\bar{k}),x(\bar{k}))$ at large $z(\bar{k})$ means that in an integration by parts we could not drop surface terms, with the presence of such surface terms preventing Hermiticity or self-adjointness. With the eigenstates of the Hamiltonian not being normalizable, there not only are negative norm states present, they are infinitely negative.

Surprisingly, it is this very inability to drop surface terms in an integration by parts that actually saves the theory (Bender and Mannheim 2008). Specifically, we have seen that we are working with a Hamiltonian H_S (and likewise $H_{\rm PU}$) that is not Hermitian. However, all the energy eigenvalues associated with H_S and $H_{\rm PU}$ are real. Now Hermiticity is only **sufficient** for real eigenvalues, with the **necessary** condition (Bender and Mannheim 2012, Mannheim 2018) being that the Hamiltonian have an antilinear symmetry. The theory thus falls into the class of PT theories (P is the linear parity operator and T is the antilinear time reversal operator) developed by Bender and collaborators (Bender 2007, Bender 2019).

Critical to the PT program is that the wave functions be normalizable in some domain in the complex plane, a domain known technically as a Stokes wedge. Since the wave functions are not normalizable with real z or real $z(\bar{k})$, we have to continue z and $z(\bar{k})$ into the complex plane in order to make them normalizable. Then the theory is well-defined, with, as we discuss below, the domain of the measure needed for the path integral accordingly also having to be continued into the complex plane in order to make it be well-defined too (Bender and Mannheim 2008, Mannheim 2018). For the particular case of $\psi_0(z,x)$ and $\psi_0(z(\bar{k}),x(\bar{k}))$, replacing z by $-iz$ and $z(\bar{k})$ by $-iz(\bar{k})$ would then make both $\psi_0(z,x)$ and $\psi_0(z(\bar{k}),x(\bar{k}))$ normalizable. (We have no need to modify x or $x(\bar{k})$ since the wave functions already are well behaved when these quantities become large.)

To achieve the continuation of z or $z(\bar{k})$ at the level of operators we effect similarity transformations as they preserve both energy eigenvalues and canonical commutators. We introduce

$$S({\rm PU}) = e^{\pi p_z z/2}, \qquad S(S) = e^{\pi \int d^3x \pi_0(\bar{x},t)\phi(\bar{x},t)/2}, \tag{7.1}$$

and obtain

$$\begin{aligned} &S({\rm PU})zS({\rm PU})^{-1} = -iz \equiv y, \qquad S({\rm PU})p_zS({\rm PU})^{-1} = ip_z \equiv q, \\ &S(S)z(\bar{k})S(S)^{-1} = -iz(\bar{k}) \equiv y(\bar{k}), \qquad S(S)p_z(\bar{k})S(S)^{-1} = ip_z(\bar{k}) \equiv q(\bar{k}). \end{aligned} \tag{7.2}$$

7.1 The PU case

For the PU oscillator this leads to

$$
\begin{aligned}
S(\mathrm{PU})H_{\mathrm{PU}}S(\mathrm{PU})^{-1} = \bar{H}_{\mathrm{PU}} = \tfrac{1}{2}p_x^2(t) - iq(t)x(t) + \tfrac{1}{2}\left(\omega_1^2+\omega_2^2\right)x^2(t) + \tfrac{1}{2}\omega_1^2\omega_2^2 y^2(t),\\
[y(t),q(t)] = i, \qquad [x(t),p_x(t)] = i.
\end{aligned}
\tag{7.3}
$$

With p and q being taken to be PT even and y and x being taken to be PT odd (Bender and Mannheim 2008), the PT invariance of $\bar{H}_{\mathrm{PU}}$ and of the $[y,q]=i$ and $[x,p_x]=i$ commutators follows. Now when a Hamiltonian is not Hermitian the action of it to the right and the action of it to the left are not related by Hermitian conjugation. Thus in general one must distinguish between right and left eigenstates, both for the vacuum and the states that can be excited out of it. Thus we represent the $[y,q]=i$ and $[x,p_x]=i$ commutators by $q=-i\overrightarrow{\partial_y}$, $p_x=-i\overrightarrow{\partial_x}$ when acting to the right, and by $q=i\overleftarrow{\partial_y}$, $p_x=i\overleftarrow{\partial_x}$ when acting to the left. This then leads to right and left ground state wave functions of the form (Bender and Mannheim 2008)

$$
\begin{aligned}
\psi_0^R(y,x) &= \exp[-\tfrac{1}{2}(\omega_1+\omega_2)\omega_1\omega_2 y^2 - \omega_1\omega_2 yx - \tfrac{1}{2}(\omega_1+\omega_2)x^2],\\
\psi_0^L(y,x) &= \exp[-\tfrac{1}{2}(\omega_1+\omega_2)\omega_1\omega_2 y^2 + \omega_1\omega_2 yx - \tfrac{1}{2}(\omega_1+\omega_2)x^2],.
\end{aligned}
\tag{7.4}
$$

Given these wave functions the vacuum normalization is given by (Bender and Mannheim 2008)

$$
\begin{aligned}
\langle\Omega^L|\Omega^R\rangle &= \int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dx\langle\Omega^L|y,x\rangle\langle y,x|\Omega^R\rangle = \int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dx\psi_0^L(y,x)\psi_0^R(y,x)\\
&= \int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dx\exp[-(\omega_1+\omega_2)\omega_1\omega_2 y^2 - (\omega_1+\omega_2)x^2] = \frac{\pi}{(\omega_1\omega_2)^{1/2}(\omega_1+\omega_2)},
\end{aligned}
\tag{7.5}
$$

with the vacuum state thus being normalizable. In the following we shall understand the wave functions to have been normalized to one, so that $\int dydx\psi_0^L(y,x)\psi_0^R(y,x)=1$ and $\langle\Omega^L|\Omega^R\rangle=1$.

With the above PT assignments and with $\dot{y} = i[\bar{H}_{\rm PU}, y] = -ix$, $\dot{x} = p_x$, $\dot{p}_x = iq - (\omega_1^2 + \omega_2^2)x$, $\dot{q} = -\omega_1^2\omega_2^2 y$, we set

$$\begin{aligned}
y(t) &= -ia_1 e^{-i\omega_1 t} + a_2 e^{-i\omega_2 t} - i\hat{a}_1 e^{i\omega_1 t} + \hat{a}_2 e^{i\omega_2 t}, \\
x(t) &= -i\omega_1 a_1 e^{-i\omega_1 t} + \omega_2 a_2 e^{-i\omega_2 t} + i\omega_1 \hat{a}_1 e^{i\omega_1 t} - \omega_2 \hat{a}_2 e^{i\omega_2 t}, \\
p_x(t) &= -\omega_1^2 a_1 e^{-i\omega_1 t} - i\omega_2^2 a_2 e^{-i\omega_2 t} - \omega_1^2 \hat{a}_1 e^{i\omega_1 t} - i\omega_2^2 \hat{a}_2 e^{i\omega_2 t}, \\
q(t) &= \omega_1\omega_2[-\omega_2 a_1 e^{-i\omega_1 t} - i\omega_1 a_2 e^{-i\omega_2 t} + \omega_2 \hat{a}_1 e^{i\omega_1 t} + i\omega_1 \hat{a}_2 e^{i\omega_2 t}], \\
a_1 e^{-i\omega_1 t} &= \frac{1}{2(\omega_1^2 - \omega_2^2)}\left[-i\omega_2^2 y(t) - p_x(t) + i\omega_1 x(t) + \frac{q(t)}{\omega_1}\right], \\
\hat{a}_1 e^{+i\omega_1 t} &= \frac{1}{2(\omega_1^2 - \omega_2^2)}\left[-i\omega_2^2 y(t) - p_x(t) - i\omega_1 x(t) - \frac{q(t)}{\omega_1}\right], \\
ia_2 e^{-i\omega_2 t} &= \frac{1}{2(\omega_1^2 - \omega_2^2)}\left[i\omega_1^2 y(t) + p_x(t) - i\omega_2 x(t) - \frac{q(t)}{\omega_2}\right], \\
i\hat{a}_2 e^{+i\omega_2 t} &= \frac{1}{2(\omega_1^2 - \omega_2^2)}\left[i\omega_1^2 y(t) + p_x(t) + i\omega_2 x(t) + \frac{q(t)}{\omega_2}\right].
\end{aligned} \tag{7.6}$$

In (7.6) we have introduced a_1, a_2, $\hat{a}_1$ and $\hat{a}_2$, with the four creation and annihilation operators obeying $PTa_1TP = a_1$, $PTa_2TP = -a_2$, $PT\hat{a}_1TP = \hat{a}_1$, $PT\hat{a}_2TP = -\hat{a}_2$, so as to enforce the PT assignments of y, x, p_x and q. Comparing with (5.5) we have $(a_1, a_2, a_1^\dagger, a_2^\dagger) \to (a_1, ia_2, \hat{a}_1, i\hat{a}_2)$.

With (7.3) and (7.6) the Hamiltonian is given by

$$\bar{H}_{\rm PU}=2(\omega_1^2-\omega_2^2)\left(\omega_1^2\hat{a}_1a_1+\omega_2^2\hat{a}_2a_2\right)+\tfrac{1}{2}(\omega_1+\omega_2), \tag{7.7}$$

and the operator commutation algebra is given by

$$\begin{aligned}
&[a_1,\hat{a}_1]=\frac{1}{2\omega_1(\omega_1^2-\omega_2^2)},\quad [a_2,\hat{a}_2]=\frac{1}{2\omega_2(\omega_1^2-\omega_2^2)},\\
&[a_1,a_2]=0,\quad [a_1,\hat{a}_2]=0,\quad [\hat{a}_1,a_2]=0,\quad [\hat{a}_1,\hat{a}_2]=0.
\end{aligned} \tag{7.8}$$

With the PT assignments of a_1, a_2, $\hat{a}_1$ and $\hat{a}_2$, we confirm the PT invariance of (7.7) and (7.8). In (7.7) and (7.8) the relative signs are all positive (we take $\omega_1>\omega_2>0$ for definitiveness), so these equations define a standard positive energy, positive norm, two-dimensional harmonic oscillator system. Given the creation and annihilation operators the left and right vacua are defined by

$$\langle\Omega^L|\hat{a}_1=0,\qquad \langle\Omega^L|\hat{a}_2=0,\qquad a_1|\Omega^R\rangle=0,\qquad a_2|\Omega^R\rangle=0. \tag{7.9}$$

By exciting modes out of the left and right vacua we can build excited states that have positive norm (Bender and Mannheim 2008), viz. $\langle n^L|m^R\rangle=\delta_{nm}$, and obey a completeness relation

$$\sum|n_1^R\rangle\langle n_1^L|+\sum|n_2^R\rangle\langle n_2^L|=I. \tag{7.10}$$

Even though these norms are all positive, the insertion of (7.10) into $-i\langle\Omega^L|T[y(t)y(0)]|\Omega^R\rangle$ (corresponding to $+i\langle\Omega^L|T[z(t)z(0)]|\Omega^R\rangle$) generates the relative minus sign in the nonrelativistic limit of the $-[1/(k^2-M_1^2)-1/(k^2-M_2^2)]/(M_1^2-M_2^2)$ propagator given in (4.2), viz. $-[1/(\omega^2-\omega_1^2)-1/(\omega^2-\omega_2^2)]/(\omega_1^2-\omega_2^2)$. We thus establish the consistency and physical viability of the similarity transformed PU oscillator theory.

7.2 The relativistic case

For $S(\mathcal{S})H_S S(\mathcal{S})^{-1}=\bar{H}_S$ we introduce creation and annihilation operators for $\bar{\phi}=S(\mathcal{S})\phi S(\mathcal{S})^{-1}=-i\phi(x)$ of the form

$$\bar{\phi}(x)=\int\frac{d^3k}{(2\pi)^{3/2}}\left[-ia_1(\bar{k})e^{-i\omega_1(\bar{k})t+i\bar{k}\cdot\bar{x}}+a_2(\bar{k})e^{-i\omega_2(\bar{k})t+i\bar{k}\cdot\bar{x}}-i\hat{a}_1(\bar{k})e^{i\omega_1(\bar{k})t-i\bar{k}\cdot\bar{x}}+\hat{a}_2(\bar{k})e^{i\omega_2(\bar{k})t-i\bar{k}\cdot\bar{x}}\right]. \tag{7.11}$$

Comparing with (4.5) we have $(a_1(\bar{k}),a_2(\bar{k}),a_1^\dagger(\bar{k}),a_2^\dagger(\bar{k}))\to(a_1(\bar{k}),ia_2(\bar{k}),\hat{a}_1(\bar{k}),i\hat{a}_2(\bar{k}))$. Like $y(t)$, $\bar{\phi}(x)$ is PT odd. The PT even Hamiltonian and PT-preserving commutation relations are given by (Bender and Mannheim 2008)

$$\begin{aligned}S(\mathcal{S})H_S S(\mathcal{S})^{-1}=\bar{H}_S\;&=\;\frac{1}{2}\int d^3k\Big[2(M_1^2-M_2^2)(\bar{k}^2+M_1^2)\left[\hat{a}_1(\bar{k})a_1(\bar{k})+a_1(\bar{k})\hat{a}_1(\bar{k})\right]\\&+\;2(M_1^2-M_2^2)(\bar{k}^2+M_2^2)\left[\hat{a}_2(\bar{k})a_2(\bar{k})+a_2(\bar{k})\hat{a}_2(\bar{k})\right]\Big],\end{aligned} \tag{7.12}$$

and

$$\begin{aligned}&[\dot{\bar{\phi}}(\bar{x},t),\bar{\phi}(0)]=0,\qquad[\ddot{\bar{\phi}}(\bar{x},t),\bar{\phi}(0)]=0,\qquad[\dddot{\bar{\phi}}(\bar{x},t),\bar{\phi}(0)]=i\delta^3(x),\\&[a_1(\bar{k}),\hat{a}_1(\bar{k}')]=[2(M_1^2-M_2^2)(\bar{k}^2+M_1^2)^{1/2}]^{-1}\delta^3(\bar{k}-\bar{k}'),\\&[a_2(\bar{k}),\hat{a}_2(\bar{k}')]=[2(M_1^2-M_2^2)(\bar{k}^2+M_2^2)^{1/2}]^{-1}\delta^3(\bar{k}-\bar{k}'),\\&[a_1(\bar{k}),a_2(\bar{k}')]=0,\quad[a_1(\bar{k}),\hat{a}_2(\bar{k}')]=0,\quad[\hat{a}_1(\bar{k}),a_2(\bar{k}')]=0,\quad[\hat{a}_1(\bar{k}),\hat{a}_2(\bar{k}')]=0.\end{aligned} \tag{7.13}$$

With all relative signs being positive (we take $M_1^2>M_2^2$ for definitiveness), there are no states of negative norm or of negative energy. The discussion completely parallels that of the PU oscillator model given above.

We introduce

$$
\begin{aligned}
y(\bar{k},t) &= -ia_1(\bar{k})e^{-i\omega_1(\bar{k})t} + a_2(\bar{k})e^{-i\omega_2(\bar{k})t} - i\hat{a}_1(\bar{k})e^{i\omega_1(\bar{k})t} + \hat{a}_2(\bar{k})e^{i\omega_2(\bar{k})t}, \\
x(\bar{k},t) &= -i\omega_1(\bar{k})a_1(\bar{k})e^{-i\omega_1(\bar{k})t} + \omega_2(\bar{k})a_2(\bar{k})e^{-i\omega_2(\bar{k})t} + i\omega_1(\bar{k})\hat{a}_1(\bar{k})e^{i\omega_1(\bar{k})t} - \omega_2(\bar{k})\hat{a}_2(\bar{k})e^{i\omega_2(\bar{k})t}, \\
p_x(\bar{k},t) &= -\omega_1^2(\bar{k})a_1(\bar{k})e^{-i\omega_1(\bar{k})t} - i\omega_2^2(\bar{k})a_2(\bar{k})e^{-i\omega_2(\bar{k})t} - \omega_1^2(\bar{k})\hat{a}_1(\bar{k})e^{i\omega_1(\bar{k})t} - i\omega_2^2(\bar{k})\hat{a}_2(\bar{k})e^{i\omega_2(\bar{k})t}, \\
q(\bar{k},t) &= \omega_1(\bar{k})\omega_2(\bar{k})[-\omega_2(\bar{k})a_1(\bar{k})e^{-i\omega_1(\bar{k})t} - i\omega_1(\bar{k})a_2(\bar{k})e^{-i\omega_2(\bar{k})t} + \omega_2(\bar{k})\hat{a}_1(\bar{k})e^{i\omega_1(\bar{k})t} + i\omega_1(\bar{k})\hat{a}_2(\bar{k})e^{i\omega_2(\bar{k})t}],
\end{aligned}
\tag{7.14}
$$

with $\bar{H}_S$ then taking the form

$$
\bar{H}_S = \int d^3k \left[\frac{p_x^2(\bar{k},t)}{2} - iq(\bar{k},t)x(\bar{k},t) + \frac{1}{2}\left[\omega_1^2(\bar{k}) + \omega_2^2(\bar{k})\right]x^2(\bar{k},t) + \frac{1}{2}\omega_1^2(\bar{k})\omega_2^2(\bar{k})y^2(\bar{k},t)\right]. \tag{7.15}
$$

Introducing left and right vacua that obey

$$
\langle\Omega^L|\hat{a}_1(\bar{k}) = 0, \qquad \langle\Omega^L|\hat{a}_2(\bar{k}) = 0, \qquad a_1(\bar{k})|\Omega^R\rangle = 0, \qquad a_2(\bar{k})|\Omega^R\rangle = 0 \tag{7.16}
$$

for all $\bar{k}$, we find that

$$
\begin{aligned}
\langle\Omega^L|\bar{H}_S|\Omega^R\rangle &= \int d^3k\left[\frac{1}{2}(\bar{k}^2 + M_1^2)^{1/2} + \frac{1}{2}(\bar{k}^2 + M_2^2)^{1/2}\right]\delta^3(0), \\
\langle\Omega^L|\Omega^R\rangle &= \Pi_{\bar{k}}\int_{-\infty}^{\infty} dy(\bar{k})\int_{-\infty}^{\infty} dx(\bar{k})\langle\Omega^L|y(\bar{k}),x(\bar{k})\rangle\langle y(\bar{k}),x(\bar{k})|\Omega^R\rangle \\
&= \Pi_{\bar{k}}\int_{-\infty}^{\infty} dy(\bar{k})\int_{-\infty}^{\infty} dx(\bar{k})\psi_0^L(y(\bar{k}),x(\bar{k}))\psi_0^R(y(\bar{k}),x(\bar{k})) = \Pi_{\bar{k}}1 = 1.
\end{aligned}
\tag{7.17}
$$

We thus confirm that the vacuum normalization is both finite and positive, while the vacuum energy has the conventional zero-point infinity associated with an infinite number of modes. (This infinity occurs because $\bar{H}_S$ contains an infinite number of modes and not because $\langle\Omega^L|\Omega^R\rangle$ itself is infinite.) We thus establish the consistency and physical viability of the similarity transformed higher-derivative scalar field theory. And we note that even though all the norms are positive, the insertion of (7.10) into $-i\langle\Omega^L|T[\bar{\phi}(x)\bar{\phi}(0)]|\Omega^R\rangle$ (corresponding to $+i\langle\Omega^L|T[\phi(x)\phi(0)]|\Omega^R\rangle$) generates the relative minus sign in $-[1/(k^2-M_1^2)-1/(k^2-M_2^2)]/(M_1^2-M_2^2)$ (Bender and Mannheim 2008). Thus with one similarity transform into an appropriate Stokes wedge we solve both the vacuum normalization problem and the negative norm problem.

At this point we can see the key aspect of our study. Ordinarily in quantum field theory it is taken as a given that one should use the Dirac inner product $\langle\Omega|\Omega\rangle$, viz. $\langle\Omega^R|\Omega^R\rangle$, for the vacuum. And also it is taken as a given that this inner product is finite. In this paper we have provided a procedure for checking whether this is in fact the case, and presented a second-order plus fourth-order derivative model in which it explicitly is not finite. For this particular model we have found a different inner product, viz. $\langle\Omega^L|\Omega^R\rangle$, that is finite. (For a Hamiltonian that is Hermitian $|\Omega^R\rangle=|\Omega\rangle$, $\langle\Omega^L|=\langle\Omega|$.) And thus in general one has to determine whether or not $\langle\Omega^R|\Omega^R\rangle$ is finite on case by case basis. We now discuss our findings from the perspective of path integrals.

8 Path integrals and the normalization of the vacuum

For the PU oscillator the Minkowski path integral (PI) associated with $I_{\rm PU}$ is of the form

$$PI(MINK) = \int D[z]D[dz/dt]\exp\left[\frac{i}{2}\int_{-\infty}^{\infty}dt\left(\left(\frac{d^2z}{dt^2}\right)^2 - \left(\omega_1^2+\omega_2^2\right)\left(\frac{dz}{dt}\right)^2 + \omega_1^2\omega_2^2z^2\right)\right]. \tag{8.1}$$

Since the theory is fourth order we need four pieces of information to solve the equations of motion. The pieces that are the most convenient for path integral purposes are two initial and two final conditions, hence the path integral measure is over both z and dz/dt. However, we had noted above that the PU theory is a constrained theory. Thus we must treat the z and dz/dt path integrations as independent. We can do this directly as shown in the measure in (8.1), or replace (8.1) by

$$PI(MINK, x, z) = \int D[z]D[x]\exp\left[\frac{i}{2}\int_{-\infty}^{\infty}dt\left(\left(\frac{dx}{dt}\right)^2 - \left(\omega_1^2+\omega_2^2\right)x^2 + \omega_1^2\omega_2^2z^2\right)\right]. \tag{8.2}$$

To make the path integral converge rather than just oscillate we first use the Feynman $i\epsilon$ prescription and replace ω_1^2 and ω_2^2 by $\omega_1^2 - i\epsilon$ and $\omega_2^2 - i\epsilon$. This yields

$$PI(MINK, x, z) = \int D[z]D[x]\exp\left[\frac{1}{2}\int_{-\infty}^{\infty}dt\left(i\left(\frac{dx}{dt}\right)^2 - i\left(\omega_1^2+\omega_2^2\right)x^2 + i\omega_1^2\omega_2^2z^2 - 2\epsilon x^2 + \epsilon\left(\omega_1^2+\omega_2^2\right)z^2\right)\right], \tag{8.3}$$

as integrated over paths with real x and real z. However, while the x^2 term is now damped the z^2 term is not. Consequently, as integrated with a real measure the path integral does not exist. Now the path integral is used to generate time-ordered Green's functions such as $D(x) = i\langle\Omega|T[\phi(x)\phi(0)]|\Omega\rangle$ (hence the $i\epsilon$ prescription). And thus these Green's functions will not be finite, with the vacuum in which the Green's function matrix elements are evaluated thus not being normalizable.

Study of the Minkowski path integral thus gives us an alternate way to determine whether or not $\langle\Omega|\Omega\rangle$ is finite: the path integral with a real measure either exists or does not exist.

To make (8.2) exist we need to damp the z^2 term, but not modify the x^2 term. Thus we continue z into the complex plane and replace it by $y = -iz$, while leaving x real. The path integral for Minkowski time then takes the form

$$PI(MINK, x, y) = \int D[y]D[x]\exp\left[\frac{1}{2}\int_{-\infty}^{\infty}dt\left(i\left(\frac{dx}{dt}\right)^2 - i\left(\omega_1^2+\omega_2^2\right)x^2 - i\omega_1^2\omega_2^2y^2 - 2\epsilon x^2 - \epsilon\left(\omega_1^2+\omega_2^2\right)y^2\right)\right]. \tag{8.4}$$

This puts us into a domain in the complex plane (known as a Stokes wedge) in which the path integral is now fully defined, and now the vacuum state is normalizable. This completely parallels the discussion of $\psi_0(z, x)$ that we gave above.

Now for Euclidean.

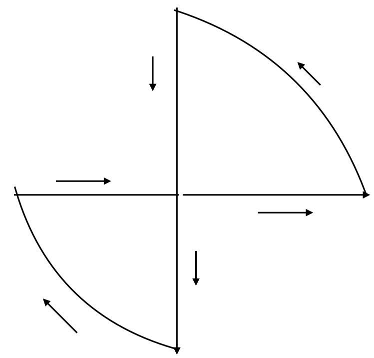

Figure 1: Wick contour

Our concerns here could be missed in a Euclidean time path integral approach. Specifically, if we disperse in t (assuming of course that we can, i.e., that the Cauchy-Riemann equations for complex t are obeyed), we can write

$$\int_{-\infty}^{\infty}+\int_{\infty}^{i\infty}+\int_{i\infty}^{-i\infty}+\int_{-i\infty}^{-\infty}=\text{pole terms plus cut contributions}, \tag{8.5}$$

i.e., along the real axis, then upper-half-plane quarter circle, then down the imaginary axis, and then lower-half-plane quarter circle. Assuming no pole, cut or circle contributions, and on setting $\tau = it$ and letting I denote the action, from (8.1) and (8.2) we obtain

$$\begin{aligned} I(MINK,z,x) &\equiv \int_{-\infty}^{\infty} dt \equiv -\int_{i\infty}^{-i\infty} dt \equiv I(EUCL,z,x), \\ PI(EUCL,z,x) &= \int D[z]D[dz/d\tau]\exp\left[-\frac{1}{2}\int_{-\infty}^{\infty} d\tau\left(\left(\frac{d^2z}{d\tau^2}\right)^2+\left(\omega_1^2+\omega_2^2\right)\left(\frac{dz}{d\tau}\right)^2+\omega_1^2\omega_2^2z^2\right)\right] \\ &= \int D[z]D[x]\exp\left[-\frac{1}{2}\int_{-\infty}^{\infty} d\tau\left(\left(\frac{dx}{d\tau}\right)^2+\left(\omega_1^2+\omega_2^2\right)x^2+\omega_1^2\omega_2^2z^2\right)\right]. \end{aligned} \tag{8.6}$$

Given the overall minus sign that multiplies the Euclidean action on every path, we see that with real z and real $x = dz/d\tau$ the Euclidean path integral is well behaved. The same is true of the analog relativistic second-order plus fourth-order scalar field theory path integral (Hawking and Hertog 2002), However, the Minkowski time path integral with a real measure is not.

Thus we conclude that the pole and/or cut and/or circle contributions are not only not ignorable, they generate an infinite contribution. Hence their contribution in a Wick rotation cannot be ignored and the Euclidean time path integral does not correctly describe the situation.

In parallel, if we set $y = -iz$, then (8.6) is replaced by

$$
\begin{aligned}
&I(MINK, y, x) \equiv I(EUCL, y, x),\\
&PI(EUCL, y, x) = \int D[y]D[x]\exp\left[\frac{1}{2}\int_{-\infty}^{\infty} d\tau\left(-\left(\frac{dx}{d\tau}\right)^2 - \left(\omega_1^2+\omega_2^2\right)x^2 + \omega_1^2\omega_2^2 y^2\right)\right].
\end{aligned} \tag{8.7}
$$

And since the y and x path integrations are independent, now it is the Euclidean time path integral that is not well defined.

Thus with either z or $y = -iz$, in neither case are the Minkowski time and Euclidean time path integrals simultaneously finite.

These same remarks carry over directly to the field theory case, **and thus we see that even if finite, a Euclidean time path integral approach is only valid if the vacuum state of the theory (as determined in a Minkowski time analysis) is normalizable. Otherwise the Wick contour rotation fails.**

9 Interactions

In developing Wick's contraction theorem in quantum field theory one needs to put the time-ordered product of Heisenberg fields $\phi(x)$, viz.

$$\tau(x_1, ..., x_n) = \langle\Omega|T[\phi(x_1)...\phi(x_n)]|\Omega\rangle, \tag{9.1}$$

into a form that can be developed perturbatively. To this end one introduces a set of in-fields $\phi_{in}(x)$ that satisfy free field equations with Hamiltonian H_{in}. And one also introduces an evolution operator $U(t)$ that evolves with the interaction Hamiltonian $H_I(t)$ according to

$$i\frac{\partial U(t)}{\partial t} = H_I(t)U(t). \tag{9.2}$$

With this $U(t)$ we can relate $\phi(x)$ and $\phi_{in}(x)$ according to

$$\phi(\bar{x}, t) = U^{-1}(t)\phi_{in}(\bar{x}, t)U(t). \tag{9.3}$$

If one introduces $U(t, t') = U(t)U^{-1}(t')$, then $U(t, t')$ is given by

$$U(t, t') = 1 - i\int_{t'}^{t} dt_1 H_I(t_1)U(t_1, t') = T\left[\exp\left(-i\int_{t'}^{t} dt_1 H_I(t_1)\right)\right]. \tag{9.4}$$

Using these relations we obtain (see e.g. Bjorken and Drell 1965) viz.

$$\tau(x_1, ..., x_n) = \langle\Omega|U^{-1}(t)T\left[\phi_{in}(x_1)...\phi_{in}(x_n)\exp\left(-i\int_{-t}^{t} dt_1 H_I(t_1)\right)\right]U(-t)|\Omega\rangle. \tag{9.5}$$

The contributions due to the $U(t)|\Omega\rangle$ and $\langle\Omega|U^{-1}(t)$ terms lead to

$$\tau(x_1, ..., x_n) = \langle\Omega|T\left[\phi_{in}(x_1)...\phi_{in}(x_n)\exp\left(-i\int_{-t}^{t} dt_1 H_I(t_1)\right)\right]|\Omega\rangle \times \langle\Omega|T\left[\exp\left(i\int_{-t}^{t} dt_1 H_I(t_1)\right)\right]|\Omega\rangle. \tag{9.6}$$

After inverting the last term we obtain the standard form (Bjorken and Drell 1965)

$$\tau(x_1,...,x_n)=\frac{\langle\Omega|T\left[\phi_{in}(x_1)...\phi_{in}(x_n)\exp\left(-i\int_{-t}^{t}dt_1H_I(t_1)\right)\right]|\Omega\rangle}{\langle\Omega|T\left[\exp\left(-i\int_{-t}^{t}dt_1H_I(t_1)\right)\right]|\Omega\rangle}. \tag{9.7}$$

If one starts with (9.7) it would appear that the normalization of the vacuum state is actually irrelevant since it would drop out of the ratio. And so it would not appear to matter if it did happen to be infinite.

However, this is not the case since we could only go from (9.6) to (9.7) if $\langle\Omega|T\left[\exp\left(i\int_{-t}^{t}dt_1H_I(t_1)\right)\right]|\Omega\rangle$ is finite. And it would not be if the vacuum state is not normalizable.

If we expand $\langle\Omega|T\left[\exp\left(i\int_{-t}^{t}dt_1H_I(t_1)\right)\right]|\Omega\rangle$ out as a power series in H_I the first term is $\langle\Omega|\Omega\rangle$ as calculated in a free theory. Thus, as we had noted above, for finiteness we need this term to be finite and need the power series expansion in H_I to be renormalizable in order for $\langle\Omega|T\left[\exp\left(i\int_{-t}^{t}dt_1H_I(t_1)\right)\right]|\Omega\rangle$ to be finite.

However, for a nonnormalizable vacuum the standard Wick expansion and Feynman rules are not valid. Since this concern is of relevance to radiative corrections to Einstein gravity we return to this point below.

10 Fermions

For fermions we have to deal with anticommutators such as

$$bb^\dagger + b^\dagger b = 1. \tag{10.1}$$

Also, because of the Pauli principle we have

$$b^2 = 0, \qquad b^{\dagger 2} = 0. \tag{10.2}$$

We can represent (10.1) and (10.2) by matrices of the form

$$b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad b^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{10.3}$$

Thus, unlike the infinite-dimensional matrix representation of the bosonic a and $a^\dagger$ that obey $aa^\dagger - a^\dagger a = 1$, the fermionic b and $b^\dagger$ matrices are finite dimensional. Thus with a finite number of degrees of freedom, the fermion vacuum that obeys $b|\Omega\rangle = 0$ has a finite $\langle\Omega|\Omega\rangle$ norm.

11 Implications for radiative corrections in quantum Einstein gravity

As a quantum theory the standard second-order derivative Einstein gravitational theory is not renormalizable. Since graviton loops generate higher-derivative gravity terms, one can construct a candidate theory of quantum gravity by augmenting the Einstein Ricci scalar action with a term that is quadratic in the Ricci scalar. This gives a quantum gravity action of the generic form

$$I_{\rm GRAV} = \int d^4x(-g)^{1/2}\left[6M^2R^{\alpha}{}_{\alpha} + (R^{\alpha}{}_{\alpha})^2\right]. \tag{11.1}$$

On adding on a matter source with energy-momentum tensor $T_{\mu\nu}$, variation of this action with respect to the metric generates a gravitational equation of motion of the form

$$-6M^2G^{\mu\nu} + V^{\mu\nu} = -\frac{1}{2}T^{\mu\nu}. \tag{11.2}$$

Here $G_{\mu\nu}$ and $V_{\mu\nu}$ are of the form

$$\begin{aligned} G^{\mu\nu} &= R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}R_{\alpha\beta}, \\ V^{\mu\nu} &= 2g^{\mu\nu}\nabla_\beta\nabla^\beta R^{\alpha}{}_{\alpha} - 2\nabla^\nu\nabla^\mu R^{\alpha}{}_{\alpha} - 2R^{\alpha}{}_{\alpha}R^{\mu\nu} + \frac{1}{2}g^{\mu\nu}(R^{\alpha}{}_{\alpha})^2. \end{aligned} \tag{11.3}$$

If we now linearize about flat spacetime with background metric $\eta_{\mu\nu}$ and fluctuation metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, to first perturbative order we obtain

$$\begin{aligned} \delta G_{\mu\nu} &= \frac{1}{2}\left(\partial_\alpha\partial^\alpha h_{\mu\nu} - \partial_\mu\partial^\alpha h_{\alpha\nu} - \partial_\nu\partial^\alpha h_{\alpha\mu} + \partial_\mu\partial_\nu h\right) - \frac{1}{2}\eta_{\mu\nu}\left(\partial_\alpha\partial^\alpha h - \partial^\alpha\partial^\beta h_{\alpha\beta}\right), \\ \delta V_{\mu\nu} &= [2\eta_{\mu\nu}\partial_\alpha\partial^\alpha - 2\partial_\mu\partial_\nu][\partial_\beta\partial^\beta h - \partial_\lambda\partial_\kappa h^{\lambda\kappa}], \end{aligned} \tag{11.4}$$

where $h = \eta^{\mu\nu}h_{\mu\nu}$. On taking the trace of the fluctuation around a background (11.2) we obtain

$$[M^2 + \partial_\beta\partial^\beta]\left(\partial_\lambda\partial^\lambda h - \partial_\kappa\partial_\lambda h^{\kappa\lambda}\right) = -\frac{1}{12}\eta^{\mu\nu}\delta T_{\mu\nu}. \tag{11.5}$$

In the convenient transverse gauge where $\partial_\mu h^{\mu\nu} = 0$, the propagator for h is given by

$$D(h, k^2) = -\frac{1}{k^2(k^2 - M^2)} = \frac{1}{M^2}\left(\frac{1}{k^2} - \frac{1}{k^2 - M^2}\right). \tag{11.6}$$

As we see, in this case the $1/k^2$ graviton propagator for h that would be associated with the Einstein tensor $\delta G_{\mu\nu}$ alone is replaced by a $D(h,k^2) = [1/k^2 - 1/(k^2 - M^2)]/M^2$ propagator. And now the leading behavior at large momenta is $-1/k^4$. In consequence, the theory is now renormalizable (Stelle 1977, Stelle 1978).

We recognize $D(h,k^2)$ as being of the same form as the second-order plus fourth-order scalar field theory propagator that was given in (4.2), with ϕ being replaced by h and with $M_1^2 = M^2$, $M_2^2 = 0$. We can thus give h an equivalent effective action of the form

$$I_h = \frac{1}{2}\int d^4x \Big[\partial_\mu\partial_\nu h \partial^\mu\partial^\nu h - M^2 \partial_\mu h \partial^\mu h\Big]. \tag{11.7}$$

The action given in (11.7) thus shares the same vacuum state normalization and negative norm challenges as the scalar field action given in (4.1).

Thus if, as is conventional, we take h to be Hermitian we would immediately encounter the negative norm problem associated with the relative minus sign in (11.6). However, since M^2 is Planck scale in magnitude, this difficulty can be postponed until observations can reach that energy scale. **However, the lack of normalizabilty of the vacuum state has consequences at all energies and cannot be postponed at all**. Specifically, with $\langle\Omega|\Omega\rangle$ being infinite we cannot even identify the propagator as $i\langle\Omega|T[h(x)h(0)]|\Omega\rangle$ since in analog to (1.8) it will obey

$$(\partial_t^2 - \bar{\nabla}^2)(\partial_t^2 - \bar{\nabla}^2 + M^2)D(h,x) = -\langle\Omega|\Omega\rangle\delta^4(x). \tag{11.8}$$

Consequently, we cannot make the standard Wick contraction expansion. And the Feynman rules that are used presupposing that $\langle\Omega|\Omega\rangle$ is finite are therefore not valid. The effective field theory approach to gravity also fails, because even at energies with $k^2 \ll M^2$ the vacuum is still not normalizable.

However, as noted above, we can resolve this concern by dropping the requirement that h be Hermitian, and set it equal to $i\bar{h}$. Then, with the theory being recognized as a PT theory, vacuum state normalization and negative norm problems are resolved, the propagator is given by $-i\langle\Omega^L|T[\bar{h}(x)\bar{h}(0)]|\Omega^R\rangle$ (corresponding to $+i\langle\Omega^L|T[h(x)h(0)]|\Omega^R\rangle$), and the theory is then consistent. In that case the only concern is that even though the M^2 field now has positive norm, it still remains in the spectrum and would eventually have to be observed.

As can be seen from (11.7), the only reason that there is an M^2 term at all is because we are considering an action that has both second-order and fourth-order terms. With a pure fourth-order theory there would be no dimensionful parameter in the action and the theory would be scale invariant. If like the gauge theories of $SU(3)\times SU(2)\times U(1)$ this scale symmetry is also local, we would be led to conformal gravity, a metric theory of gravity in which the action is left invariant under local changes of the metric of the form $g_{\mu\nu}(x)\rightarrow e^{2\alpha(x)}g_{\mu\nu}(x)$, where $\alpha(x)$ is a local function of the coordinates. The conformal gravity theory has been advocated and explored in (Mannheim 2006, Mannheim 2017) and references therein. And 't Hooft ('t Hooft 2015) has also argued that there should be an underlying local conformal symmetry in nature.

In the conformal gravity theory an action that is to be a polynomial function of the metric has the unique form

$$I_{\rm W}=-\alpha_g\int d^4x\,(-g)^{1/2}C_{\lambda\mu\nu\kappa}C^{\lambda\mu\nu\kappa}\equiv-2\alpha_g\int d^4x\,(-g)^{1/2}\left[R_{\mu\kappa}R^{\mu\kappa}-\frac{1}{3}(R^{\alpha}{}_{\alpha})^2\right], \tag{11.9}$$

where α_g is a dimensionless gravitational coupling constant, and $C_{\lambda\mu\nu\kappa}$ is the conformal Weyl tensor. The perturbative propagator has a $-1/k^4$ behavior at all k^2, and with its large k^2 behavior the theory is renormalizable (Fradkin and Tseytlin 1985). With a $-1/k^4$ propagator it would initially appear that there would be two massless particles at $k^2=0$. However, we cannot use the partial fraction decomposition given in (11.6) as a guide since its $1/M^2$ prefactor is singular in the $M^2\to 0$ limit. Because of this singular behavior the $M^2=0$ Hamiltonian becomes of nondiagonalizable Jordan-block form and only has one massless eigenstate, with the other would-be massless eigenstate becoming nonstationary (Bender and Mannheim 2008).

Thus the propagator should be constructed not as the $M^2\to 0$ limit of the ghost-like but actually ghost-free (11.6), but instead as the manifestly ghost-free limit

$$-\frac{1}{(k^2+i\epsilon)^2}=-\lim_{M^2\to 0}\frac{d}{dM^2}\left(\frac{1}{k^2-M^2+i\epsilon}\right), \tag{11.10}$$

a limit that shows that there is only one $k^2=0$ pole not two. With the Hamiltonian not being diagonalizable, it could not be Hermitian. It does however have a PT symmetry, with is ground state being normalizable. Conformal gravity is thus a fully consistent theory of quantum gravity, one which despite its fourth-order character possesses no states of negative norm, and only one massless particle, not two.

12 Final Comments

For a quantum field theory to be physically relevant it must be formulatable in a Hilbert space with an inner product that is time independent, finite and nonnegative. However, in and of itself, specifying an action and a set of canonical commutators is not enough to either fix the Hilbert space or specify the appropriate inner product. Ordinarily, one supplements these requirements with the additional (generally regarded as self-evident) requirements that the fields and the Hamiltonian of the theory be Hermitian, and that the inner product be the standard, presumed finite, Dirac $\langle n|n\rangle$ one. However, this is not automatic for any theory, and so one needs to check on a case by case basis. And we have presented a procedure for doing so. The procedure is based on using the occupation number space representation to construct an equivalent wave mechanics representation, from which we can check for the normalizability of the vacuum state, and accordingly of the states that can be excited out of it. An alternative but equivalent approach is to check whether or not the Minkowski time path integral with a real measure exists. If it does not, then the standard Dirac inner product is not finite.

Using the occupation number space representation procedure we have found a case, a second-order plus fourth-order scalar field theory, in which the standard Dirac inner product $\langle n|n\rangle$ actually is not finite. In this example the Minkowski time path integral with a real measure diverges even though the Euclidean time path integral does not. Even though contributions from the Wick rotation contour are ordinarily ignored, in this case they cannot be. Thus the use of a Euclidean time path integral can be misleading. And even if the Euclidean time path integral is well behaved, it only gives a good description of the theory if the Minkowski time path integral is well behaved too. Since $\langle\Omega|\Omega\rangle$ is not finite, use of the standard Feynman rules is not valid, with these rules not only leading to states with negative norm, they lead to states with infinite negative norm. This lack of finiteness means that the Hamiltonian is not self-adjoint when acting on these particular states.

However, the Hamiltonian of the second-order plus fourth-order scalar field theory is PT symmetric, so we can use the techniques of the PT symmetry program and continue the fields and the Hamiltonian in this theory into the complex plane. There is then a domain in the complex plane in which one can define an appropriate time-independent, positive and finite inner product, viz. the $\langle L|R\rangle$ overlap of left-eigenstates and right-eigenstates of the resulting Hamiltonian, with the resulting vacuum state then being normalizable, and with there being no states with negative or infinite $\langle L|R\rangle$ norm. In this complex domain it is the Euclidean time path integral that diverges while the Minkowski time path integral does not. So again there are contributions from the Wick rotation contour. In this complex domain the second-order plus fourth-order scalar field theory is fully consistent, unitary and renormalizable, with this analysis being relevant for the construction of a consistent quantum theory of gravity.