A distributional approach to the inverted quantum harmonic oscillator

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Organization of the talk

Part 1: A θ-depending Hamiltonian H_θ

- *Relations with Swanson...*
- *...the eigenvalues, the eigenvectors...*
- *..their properties...*
- *...and bi-coherent states.*

Part 2: the inverted quantum harmonic oscillator (IQHO)

- *The need of leaving $\mathcal{L}^2(\mathbb{R})$...*
- *...generalized eigenvectors...*
- *...and (again) bi-coherent states.*

Main reference: F. Bagarello, *A Swanson-like Hamiltonian and the inverted harmonic oscillator*, J. Phys. A, **55**. 225204 (2022)

The Swanson Hamiltonian

The non-self-adjoint Swanson Hamiltonian (Bebiano et al, 2009. There are others.) is

$$\widetilde{H}_\theta = \frac{1}{2}\left(p^2 + x^2\right) - \frac{i}{2}\left(\tan 2\theta\right)\left(p^2 - x^2\right) .$$

and acts on wave-functions in the standard Hilbert space $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$. Here

$$-\frac{\pi}{4} < \theta < \frac{\pi}{4}, \qquad x = x^\dagger, \qquad p = p^\dagger, \qquad [x, p] = i\mathbb{1}.$$

Let $c = \frac{1}{\sqrt{2}}(x + ip)$, $c^\dagger = \frac{1}{\sqrt{2}}(x - ip)$ and

$$\begin{cases} A_\theta = c\cos\theta + ic^\dagger\sin\theta = \frac{1}{\sqrt{2}}\left(e^{i\theta}x + e^{-i\theta}\,\frac{d}{dx}\right), \\ B_\theta = c^\dagger\cos\theta + ic\sin\theta = \frac{1}{\sqrt{2}}\left(e^{i\theta}x - e^{-i\theta}\,\frac{d}{dx}\right). \end{cases}$$

Then, for $\theta \neq 0$, $A_\theta^\dagger \neq B_\theta$. Moreover

$$[A_\theta, B_\theta] = \mathbb{1} \qquad [A_\theta, A_\theta^\dagger] = -[B_\theta, B_\theta^\dagger] = \mathbb{1}\cos 2\theta.$$

For these reasons, A_θ, B_θ (and their adjoints) are *pseudo-boson operators*. We have

$$\widetilde{H}_\theta = \omega_\theta\left(B_\theta\,A_\theta + \frac{1}{2}\mathbb{1}\right), \qquad \omega_\theta = \frac{1}{\cos 2\theta}$$

The Swanson Hamiltonian

In [F.B., PLA, 2010], we have found the eigenstates of $\widetilde{H}_\theta$ and $\widetilde{H}_\theta^\dagger$:

$$\begin{cases} \varphi_n^{(\theta)}(x) = \frac{1}{\sqrt{n!}}\, B_\theta^n\, \varphi_0^{(\theta)}(x) = \frac{N_\varphi}{\sqrt{2^n\, n!}}\, H_n\left(e^{i\theta}x\right)\, \exp\left(-\frac{1}{2}\, e^{2i\theta}\, x^2\right), \\ \Psi_n^{(\theta)}(x) = \frac{1}{\sqrt{n!}}\, (A_\theta^\dagger)^n\, \Psi_0^{(\theta)}(x) = \frac{N_\Psi}{\sqrt{2^n\, n!}}\, H_n\left(e^{-i\theta}x\right)\, \exp\left(-\frac{1}{2}\, e^{-2i\theta}\, x^2\right), \end{cases}$$

where $H_n(x)$ is the n-th Hermite polynomial, while $\varphi_0^{(\theta)}(x)$ and $\Psi_0^{(\theta)}(x)$ are the vacua of A_θ and $B_\theta^\dagger$:

$$A_\theta \varphi_0^{(\theta)}(x) = 0, \qquad B_\theta^\dagger \Psi_0^{(\theta)}(x) = 0.$$

Taking $\overline{N_\varphi}\, N_\Psi = \frac{e^{-i\theta}}{\sqrt{\pi}}$, we have

$$\langle \varphi_n^{(\theta)}, \Psi_m^{(\theta)} \rangle = \delta_{n,m},$$

and

$$\widetilde{H}_\theta \varphi_n^{(\theta)}(x) = E_n^{(\theta)} \varphi_n^{(\theta)}(x), \qquad \widetilde{H}_\theta^\dagger \Psi_n^{(\theta)}(x) = E_n^{(\theta)} \Psi_n^{(\theta)}(x),$$

with

$$E_n^{(\theta)} = \omega_\theta \left(n + \frac{1}{2}\right) = \frac{1}{\cos 2\theta}\left(n + \frac{1}{2}\right)$$

The Swanson Hamiltonian

Few comments:

1. The original Swanson Hamiltonian ([M.S., JMP, 2004]), looks a bit different:

$$H = \omega\left(c^\dagger c + \frac{1}{2}1\!\!1\right) + \alpha c^2 + \beta {c^\dagger}^2,$$

   where $\alpha \neq \beta$.

2. In 2020, [J. Feinberg, F.B., Ann. of Phys.] we have used bi-coherent states *attached* to A_θ and $B_\theta^\dagger$ to introduce a path integral and to compute the propagator for the Swanson model.

3. For $\theta \in \left]-\frac{\pi}{4}, \frac{\pi}{4}\right[$ and $\forall n \geq 0$, $\varphi_n^{(\theta)}(x), \Psi_n^{(\theta)}(x) \in \mathcal{L}^2(\mathbb{R})$. However, $\theta \to \pm\frac{\pi}{4}$ is a *dangerous* operation:

$$\cos(2\theta) \to 0;$$

   and

$$\Re(e^{\pm 2i\theta}) \to 0,$$

   so that $\varphi_n^{(\pm\frac{\pi}{4})}(x), \Psi_n^{(\pm\frac{\pi}{4})}(x) \notin \mathcal{L}^2(\mathbb{R})$, anymore. Also $\varphi_n^{(\theta)}(x), \Psi_n^{(\theta)}(x) \notin \mathcal{L}^2(\mathbb{R})$, if $\theta \notin \left]-\frac{\pi}{4}, \frac{\pi}{4}\right[$.

From Swanson to Swanson-like

In view of our final aim (the analysis of IQHO) we introduce here

$$H_\theta = \frac{1}{2}\left(p^2 + e^{2i\theta}\Omega^2 x^2\right),$$

for $\theta \in [-\pi, \pi]$, for the moment, and $\Omega > 0$. Then, if $\theta = \pm\frac{\pi}{2}$, H_θ becomes the Hamiltonian of the IQHO, $H_\pm = \frac{1}{2}\left(p^2 - \Omega^2 x^2\right)$.

Remark:– H_θ is very similar to $\widetilde{H}_\theta$, since this latter can be rewritten as

$$\widetilde{H}_\theta = \frac{e^{-2i\theta}}{2\cos(2\theta)}\left(p^2 + x^2 e^{4i\theta}\right),$$

which differs from H_θ because of the term $\frac{e^{-2i\theta}}{\cos(2\theta)}$, and because θ is here replaced by 2θ.

Hence it is not a surprise that we can diagonalize H_θ! For that, we use the following operators:

The Swanson-like Hamiltonian

$$A_\theta = \frac{1}{\sqrt{2\Omega}}\left(e^{i\theta/2}\Omega\, x + i\, e^{-i\theta/2} p\right), \quad B_\theta = \frac{1}{\sqrt{2\Omega}}\left(e^{i\theta/2}\Omega\, x - i\, e^{-i\theta/2} p\right),$$

$\forall\theta$. They are both densely defined: $\mathcal{S}(\mathbb{R}) \subseteq D(A_\theta)$ and $\mathcal{S}(\mathbb{R}) \subseteq D(B_\theta)$, $\forall\theta$. It is also clear that $A_\theta^\dagger \neq B_\theta$ and we have that, on $\mathcal{S}(\mathbb{R})$,

$$A_\theta^\dagger = \frac{1}{\sqrt{2\Omega}}\left(e^{-i\theta/2}\Omega\, x - i\, e^{i\theta/2} p\right), \quad B_\theta^\dagger = \frac{1}{\sqrt{2\Omega}}\left(e^{-i\theta/2}\Omega\, x + i\, e^{i\theta/2} p\right).$$

The set $\mathcal{S}(\mathbb{R})$ is stable under the action of all these operators. We have

$$A_\theta^\dagger = B_{-\theta}, \qquad B_\theta^\dagger = A_{-\theta},$$

and

$$[A_\theta, B_\theta] f(x) = f(x)$$

for all $f(x) \in \mathcal{S}(\mathbb{R})$, and for all values of θ. Hence

$$H_\theta = \Omega e^{i\theta}\left(B_\theta A_\theta + \frac{1}{2}\mathbb{1}\right), \qquad H_\theta^\dagger = \Omega e^{-i\theta}\left(A_\theta^\dagger B_\theta^\dagger + \frac{1}{2}\mathbb{1}\right) = H_{-\theta}.$$

Let us construct their eigenstates (the pseudo-bosonic way...):

The Swanson-like Hamiltonian

Step $\sharp\,1$: The vacua.
The solution of $A_\theta \varphi_0^{(\theta)}(x) = 0$ is

$$\varphi_0^{(\theta)}(x) = N^{(\theta)} e^{-\frac{1}{2}\,\Omega e^{i\theta} x^2},$$

and the other vacuum, solution of $B_\theta^\dagger \psi_0^{(\theta)}(x) = 0$, can be found as $\psi_0^{(\theta)}(x) = \varphi_0^{(-\theta)}(x)$, since $B_\theta^\dagger = A_{-\theta}$. $N^{(\pm\theta)}$ will be fixed later.

Step $\sharp\,2$: A requirement on the vacua.
As for the Swanson model, requiring that $\varphi_0^{(\theta)}(x), \psi_0^{(\theta)}(x) \in \mathcal{L}^2(\mathbb{R})$, forces $\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

Step $\sharp\,3$: The other vectors.
Using ladder operators and $H_n(y) = 2yH_{n-1}(y) - H'_{n-1}(y)$ we get:

$$\varphi_n^{(\theta)}(x) = \frac{B_\theta^n}{\sqrt{n!}}\,\varphi_0^{(\theta)}(x) = \frac{N^{(\theta)}}{\sqrt{2^n\,n!}}\,H_n\left(e^{i\theta/2}\sqrt{\Omega}\,x\right)e^{-\frac{1}{2}\,\Omega e^{i\theta} x^2}$$

and

$$\psi_n^{(\theta)}(x) = \frac{A_\theta^{\dagger\,n}}{\sqrt{n!}}\,\psi_0^{(\theta)}(x) = \varphi_n^{(-\theta)}(x) = \frac{N^{(-\theta)}}{\sqrt{2^n\,n!}}\,H_n\left(e^{-i\theta/2}\sqrt{\Omega}\,x\right)e^{-\frac{1}{2}\,\Omega e^{-i\theta} x^2}.$$

To be continued...

The Swanson-like Hamiltonian

Step $\sharp\, 4$: Properties of $\mathcal{F}_\varphi^{(\theta)} = \{\varphi_n^{(\theta)}(x)\}$ and $\mathcal{F}_\psi^{(\theta)} = \{\psi_n^{(\theta)}(x)\}$.

a. $\forall\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ and $\forall n \geq 0$, $\varphi_n^{(\theta)}(x), \psi_n^{(\theta)}(x) \in \mathcal{L}^2(\mathbb{R})$.

b. Putting $\mathbf{N}^{(\theta)} = B_\theta A_\theta$ we have

$$\begin{cases} B_\theta\, \varphi_n^{(\theta)}(x) = \sqrt{n+1}\, \varphi_{n+1}^{(\theta)}(x), & n \geq 0, \\ A_\theta\, \varphi_0^{(\theta)}(x) = 0, \qquad A_\theta\, \varphi_n^{(\theta)}(x) = \sqrt{n}\, \varphi_{n-1}^{(\theta)}(x), & n \geq 1, \\ A_\theta^\dagger\, \psi_n^{(\theta)}(x) = \sqrt{n+1}\, \psi_{n+1}^{(\theta)}(x), & n \geq 0, \\ B_\theta^\dagger\, \psi_0^{(\theta)}(x) = 0, \qquad B_\theta^\dagger\, \psi_n^{(\theta)}(x) = \sqrt{n}\, \psi_{n-1}^{(\theta)}(x), & n \geq 1, \\ \mathbf{N}^{(\theta)} \varphi_n^{(\theta)}(x) = n\, \varphi_n^{(\theta)}(x), \quad \mathbf{N}^{(\theta)\dagger} \psi_n^{(\theta)}(x) = n\, \psi_n^{(\theta)}(x), & n \geq 0, \end{cases}$$

c. We have

$$H_\theta \varphi_n^{(\theta)}(x) = E_n^{(\theta)} \varphi_n^{(\theta)}(x), \qquad H_\theta^\dagger \psi_n^{(\theta)}(x) = E_n^{(-\theta)} \psi_n^{(\theta)}(x),$$

where

$$E_n^{(\theta)} = \omega e^{i\theta} \left(n + \frac{1}{2}\right).$$

Notice that $E_n^{(-\theta)} = \overline{E_n^{(\theta)}}$, which are complex.

To be continued...

The Swanson-like Hamiltonian

Step $\sharp\, 5$: $\mathcal{F}_{\varphi}^{(\theta)}$ and $\mathcal{F}_{\psi}^{(\theta)}$ are biorthonormal.
Indeed, if we fix

$$N^{(\theta)} = \left(\frac{\Omega}{\pi}\right)^{1/4} e^{i\theta/4},$$

we can check that

$$\langle \varphi_n^{(\theta)}, \psi_m^{(\theta)} \rangle = \delta_{n,m},$$

for all $n, m \geq 0$ and for all θ.

Remark:– this follows from general facts, but a direct check is a delicate technical point, which requires subtle computations in complex contour integration.

Step $\sharp\, 6$: What if $\theta = 0$.
In this case everything collapses to the usual quantum harmonic oscillator, as it should. In this case, since $N^{(0)} = \left(\frac{\Omega}{\pi}\right)^{1/4}$,

$$\varphi_n^{(0)}(x) = \psi_n^{(0)}(x) = e_n(x) = \frac{1}{\sqrt{2^n\, n!}} \left(\frac{\Omega}{\pi}\right)^{1/4} H_n\left(\sqrt{\Omega}\, x\right) e^{-\frac{1}{2}\,\Omega\, x^2},$$

which is the well known n-th eigenstate of the quantum harmonic oscillator, as expected.

To be continued...

The Swanson-like Hamiltonian

Step $\sharp\,7$: More properties of $\mathcal{F}_\varphi^{(\theta)}$ and $\mathcal{F}_\psi^{(\theta)}$.

a. $\mathcal{F}_\varphi^{(\theta)}$ and $\mathcal{F}_\psi^{(\theta)}$ are complete in $\mathcal{L}^2(\mathbb{R})$.

b. $\mathcal{F}_\varphi^{(\theta)}$ and $\mathcal{F}_\psi^{(\theta)}$ satisfy the resolution of the identity (closure relation):

$$\sum_{n=0}^{\infty} \langle f, \varphi_n^{(\theta)} \rangle \langle \psi_n^{(\theta)}, g \rangle = \langle f, g \rangle, \tag{1}$$

$\forall f(x) \in \mathcal{L}_\psi^{(\theta)} = l.s.\{\psi_n^{(\theta)}(x)\}$ and $\forall g(x) \in \mathcal{L}_\varphi^{(\theta)} = l.s.\{\varphi_n^{(\theta)}(x)\}$

c. $\mathcal{F}_\varphi^{(\theta)}$ and $\mathcal{F}_\psi^{(\theta)}$ are **NOT** bases for $\mathcal{L}^2(\mathbb{R})$.
This is because

$$\|\varphi_n^{(\theta)}\|^2 = \|\psi_n^{(\theta)}\|^2 = \frac{1}{\sqrt{\cos(\theta)}} P_n\left(\frac{1}{\cos(\theta)}\right) \to \infty,$$

when $n \to \infty$, $\forall \theta \neq 0$. Then (1) cannot be true $\forall f(x), g(x) \in \mathcal{L}^2(\mathbb{R})$. Here $P_n(x)$ is the n-th Legendre polynomial.

The Swanson-like Hamiltonian: a similarity operator

Let us introduce an invertible operator, V_θ as follows

$$V_\theta e_n(x) = e^{i\theta/4} e_n(e^{i\theta/2}x) = \varphi_n^{(\theta)}(x),$$

$\forall n \geq 0$. This action can be extended by linearity to $\mathcal{L}_e = l.s.\{e_n(x)\}$, which is dense in $\mathcal{L}^2(\mathbb{R})$.
Of course we have $V_\theta^{-1} = V_{-\theta}$, and

$$V_\theta^{-1} e_n(x) = e^{-i\theta/4} e_n(e^{-i\theta/2}x) = \psi_n^{(\theta)}(x).$$

We have $\varphi_n^{(\theta)}(x) \in D(V_\theta^{-1})$ and $\psi_n^{(\theta)}(x) \in D(V_\theta)$, with

$$V_\theta^{-1}\varphi_n^{(\theta)}(x) = V_\theta \psi_n^{(\theta)}(x) = e_n(x).$$

One could also check that $\psi_n^{(\theta)}(x) \in D(V_\theta^\dagger)$, $\varphi_n^{(\theta)}(x) \in D(V_{-\theta}^\dagger)$, and that

$$V_{-\theta}^\dagger \varphi_n^{(\theta)}(x) = V_\theta^\dagger \psi_n^{(\theta)}(x) = e_n(x).$$

The Swanson-like Hamiltonian: a similarity operator

Remark:– Formally we write

$$V_\theta = e^{i\frac{\theta}{4}(c^2 - c^{\dagger\,2})}, \qquad [c, c^\dagger] = 1\!\!1,$$

which *looks* self-adjoint, unbounded, invertible and with unbounded inverse.

Using complex integration techniques we see that $\langle V_\theta e_n, e_m\rangle = \langle e_n, V_\theta e_m\rangle$, $\forall n, m \geq 0$, so that

$$\langle V_\theta f, g\rangle = \langle f, V_\theta g\rangle,$$

$\forall f(x), g(x) \in \mathcal{L}_e$ and $\forall \theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$. Then we have

$$\sum_{n=0}^{\infty} \langle f, \varphi_n^{(\theta)}\rangle \langle \psi_n^{(\theta)}, g\rangle = \langle f, g\rangle,$$

$\forall f(x) \in \mathcal{L}_\psi^{(\theta)}$ and $\forall g(x) \in \mathcal{L}_e$ and, similarly, that

$$\sum_{n=0}^{\infty} \langle f, \psi_n^{(\theta)}\rangle \langle \varphi_n^{(\theta)}, g\rangle = \langle f, g\rangle,$$

$\forall f(x) \in \mathcal{L}_\varphi^{(\theta)}$ and $\forall g(x) \in \mathcal{L}_e$. We have several resolutions of the identity (**none in all of $\mathcal{L}^2(\mathbb{R})$**).

The Swanson-like Hamiltonian: bicoherent states

The existence of bi-coherent states for our system is ensured by the following theorem

Assume that four strictly positive constants K_φ, K_ψ, r_φ and r_ψ exist, together with two strictly positive sequences $M_n(\varphi)$ and $M_n(\psi)$, for which

$$\lim_{n\to\infty}\frac{M_n(\varphi)}{M_{n+1}(\varphi)}=M(\varphi),\qquad \lim_{n\to\infty}\frac{M_n(\psi)}{M_{n+1}(\psi)}=M(\psi),$$

where $M(\varphi)$ and $M(\psi)$ could be infinity, and such that, for all $n\geq 0$,

$$\|\varphi_n^{(\theta)}\|\leq K_\varphi\, r_\varphi^n M_n(\varphi),\qquad \|\psi_n^{(\theta)}\|\leq K_\psi\, r_\psi^n M_n(\psi).$$

Then the series

$$\varphi^{(\theta)}(z;x)=e^{-\frac{1}{2}|z|^2}\sum_{k=0}^\infty\frac{z^k}{\sqrt{k!}}\varphi_k^{(\theta)}(x),\qquad \psi^{(\theta)}(z;x)=e^{-\frac{1}{2}|z|^2}\sum_{k=0}^\infty\frac{z^k}{\sqrt{k!}}\psi_k^{(\theta)}(x),$$

are convergent in $\mathbb{C}$. Moreover, for all $z\in\mathbb{C}$,

$$A_\theta\varphi^{(\theta)}(z;x)=z\varphi^{(\theta)}(z;x),\qquad B_\theta^\dagger\psi^{(\theta)}(z;x)=z\psi^{(\theta)}(z;x).$$

Finally we have

$$\frac{1}{\pi}\int_{\mathbb{C}}\left\langle f,\varphi^{(\theta)}\right\rangle\left\langle\psi^{(\theta)},g\right\rangle dz=\langle f,g\rangle\,,\qquad \forall f(x)\in\mathcal{L}_\psi^{(\theta)},g(x)\in\mathcal{L}_\varphi^{(\theta)}\cup\mathcal{L}_e.$$

The Swanson-like Hamiltonian: bicoherent states

The bounds on $\|\varphi_n^{(\theta)}\|$ and $\|\psi_n^{(\theta)}\|$ have been discussed before, in terms of Legendre polynomials. We find that, $\forall\theta\in I\setminus\{0\}$,

$$\|\varphi_n^{(\theta)}\|^2 \leq k^{(\theta)}\,\frac{1}{\sqrt{n}}\left(\frac{2}{\cos(\theta)}\right)^n, \qquad \|\psi_n^{(\theta)}\|^2 \leq k^{(-\theta)}\,\frac{1}{\sqrt{n}}\left(\frac{2}{\cos(\theta)}\right)^n$$

where $k^{(\theta)}$ is a θ-depending positive quantity (not so relevant here).
The series can be re-summed and we find, with a proper normalization,

$$\varphi^{(\theta)}(z;x) = \left(\frac{\Omega}{\pi}\right)^{1/4}\exp\left\{i\,\frac{\theta}{4} - z_r^2 + \sqrt{2\Omega}\,e^{i\theta/2}\,z\,x - \frac{1}{2}e^{i\theta}\Omega\,x^2\right\}$$

and

$$\psi^{(\theta)}(z;x) = \left(\frac{\Omega}{\pi}\right)^{1/4}\exp\left\{-i\,\frac{\theta}{4} - z_r^2 + \sqrt{2\Omega}\,e^{-i\theta/2}\,z\,x - \frac{1}{2}e^{-i\theta}\Omega\,x^2\right\},$$

which are both square integrable, for our allowed θ. When $\theta\to 0$ we get

$$\lim_{\theta,0}\varphi^{(\theta)}(z;x) = \lim_{\theta,0}\psi^{(\theta)}(z;x) = \left(\frac{\Omega}{\pi}\right)^{1/4}\exp\left\{-z_r^2 + \sqrt{2\Omega}\,z\,x - \frac{1}{2}\,\Omega\,x^2\right\} = \Phi(z;x),$$

the standard coherent state for the bosonic annihilation operator, with the usual normalization.

The inverted quantum oscillator: a preliminary analysis

Let

$$H = \frac{1}{2}\left(p^2 - \Omega^2 x^2\right),$$

$\Omega > 0$. H can be formally deduced by H_θ fixing θ either to $\frac{\pi}{2}$ or to $-\frac{\pi}{2}$. This suggests us to define

$$\varphi_n^{(\pm)}(x) = \varphi_n^{(\pm\frac{\pi}{2})}(x) = \frac{e^{\pm i\pi/8}}{\sqrt{2^n\, n!}}\left(\frac{\Omega}{\pi}\right)^{1/4} H_n\left(e^{\pm i\pi/4}\sqrt{\Omega}\, x\right) e^{\mp \frac{i}{2}\,\Omega\, x^2}$$

and

$$\psi_n^{(\pm)}(x) = \psi_n^{(\pm\frac{\pi}{2})}(x) = \varphi_n^{(\mp)}(x).$$

It is clear that

$$\|\varphi_n^{(\pm)}\| = \|\psi_n^{(\pm)}\| = \infty.$$

None of these functions is square-integrable. However, they are still connected to the operators $A_\pm$, $B_\pm$ and their adjoints, where

$$A_\pm = A_{\pm\frac{\pi}{2}} = \frac{1}{\sqrt{2\Omega}}\left(e^{\pm i\pi/4}\Omega\, x + i\, e^{\mp i\pi/4} p\right), \quad B_\pm = B_{\pm\frac{\pi}{2}} = \frac{1}{\sqrt{2\Omega}}\left(e^{\pm i\pi/4}\Omega\, x - i\, e^{\mp i\pi/4} p\right),$$

which are densely defined on $\mathcal{S}(\mathbb{R})$, and

$$B_\pm^\dagger = A_\mp, \qquad A_\pm^\dagger = B_\mp.$$

The inverted quantum oscillator: a preliminary analysis

We can rewrite

$$A_\pm = \frac{c \pm ic^\dagger}{\sqrt{2}}, \qquad B_\pm = \frac{c^\dagger \pm ic}{\sqrt{2}},$$

with $A_\pm^\dagger$ and $B_\pm^\dagger$ deduced as above. All these operators leave $\mathcal{S}(\mathbb{R})$ stable, and we have

$$[A_\pm, B_\pm] f(x) = f(x),$$

for all $f(x) \in \mathcal{S}(\mathbb{R})$. However, it is clear that these operators can also be applied to functions which are outside $\mathcal{S}(\mathbb{R})$, and even outside $\mathcal{L}^2(\mathbb{R})$. In fact, they act on $\varphi_n^{(\pm)}(x)$ and $\psi_n^{(\pm)}(x)$ and satisfy *standard* ladder equations:

$$\begin{cases} A_\pm\, \varphi_0^{(\pm)}(x) = 0, \qquad A_\pm\, \varphi_n^{(\pm)}(x) = \sqrt{n}\, \varphi_{n-1}^{(\pm)}(x), & n \geq 1, \\ B_\pm\, \varphi_n^{(\pm)}(x) = \sqrt{n+1}\, \varphi_{n+1}^{(\pm)}(x), & n \geq 0, \end{cases}$$

and

$$\begin{cases} B_\pm^\dagger\, \psi_0^{(\pm)}(x) = 0, \qquad B_\pm^\dagger\, \psi_n^{(\pm)}(x) = \sqrt{n}\, \psi_{n-1}^{(\pm)}(x), & n \geq 1, \\ A_\pm^\dagger\, \psi_n^{(\pm)}(x) = \sqrt{n+1}\, \psi_{n+1}^{(\pm)}(x), & n \geq 0. \end{cases}$$

The inverted quantum oscillator: a preliminary analysis

After specializing $\varphi_n^{(\theta)}(x)$, $\psi_n^{(\theta)}(x)$, A_θ and B_θ by fixing $\theta = \pm\,\frac{\pi}{2}$, let us also specialize H_θ by taking again $\theta = \pm\frac{\pi}{2}$. We put

$$H_\pm = \pm i\Omega\left(B_\pm A_\pm + \frac{1}{2}\,1\!\!1\right),$$

and we have, as expected

$$H = H_+ = H_- = H_+^\dagger,$$

at least formally. We further have

$$H_\pm \varphi_n^{(\pm)}(x) = \pm i\Omega\left(n + \frac{1}{2}\right)\varphi_n^{(\pm)}(x),$$

$\forall n \geq 0$. Hence the eigenvalues of the IQHO are purely imaginary with both a positive and a negative imaginary part.

Of course the functions $\psi_n^{(\pm)}(x)$, which are usually the eigenstates of the adjoint of the *original* Hamiltonian, are not so relevant here since the adjoint of H_+ is H_+ itself. This is in agreement with the fact that $\psi_n^{(\pm)}(x) = \varphi_n^{(\mp)}(x)$.

The inverted quantum oscillator: a distributional view

We will now show that $\varphi_n^{(\pm)}(x)$, and consequently $\psi_n^{(\pm)}(x)=\varphi_n^{(\mp)}(x)$, define tempered distributions, i.e., elements in $\mathcal{S}'(\mathbb{R})$. We will concentrate on $\varphi_n^{(+)}(x)$, here.

Step $\sharp\,1$: Introducing the linear functional.
We start introducing

$$\Phi_n^{(+)}[f]=\langle\varphi_n^{(+)},f\rangle,$$

$\forall f(x)\in\mathcal{S}(\mathbb{R})$ and $\forall n\geq 0$. Here $\langle .,.\rangle$ is the form with extend the ordinary scalar product to *compatible pairs*, i.e. to pairs of functions with are, when multiplied together, integrable (see,e.g., PIP spaces).
To check that $\Phi_n^{(+)}[f]$ is well defined, we notice that

$$\left|\Phi_n^{(+)}[f]\right|\leq\frac{(\Omega/\pi)^{1/4}}{\sqrt{2^n\,n!}}\int_{\mathbb{R}}\left|H_n\left(e^{i\pi/4}\sqrt{\Omega}\,x\right)f(x)\right|\,dx\leq M_n\sup_{x\in\mathbb{R}}(1+|x|)^{n+2}|f(x)|,$$

where, dividing and multiplying by $(1+|x|)^{n+2}$, we have introduced

$$M_n=\frac{(\Omega/\pi)^{1/4}}{\sqrt{2^n\,n!}}\int_{\mathbb{R}}\frac{|H_n(e^{i\pi/4}\sqrt{\Omega}\,x)|}{(1+|x|)^{n+2}}\,dx<\infty,$$

$\forall n$.

The inverted quantum oscillator: a distributional view

Next we observe that

$$\sup_{x\in\mathbb{R}}(1+|x|)^{n+2}|f(x)| = \sup_{x\in\mathbb{R}}\sum_{k=0}^{n+2}\left(\begin{array}{c} n+2 \\ k \end{array}\right)|x|^k|f(x)| = \sum_{k=0}^{n+2}\left(\begin{array}{c} n+2 \\ k \end{array}\right)p_{k,0}(f),$$

where $p_{k,0}(.)$ is one of the seminorms defining the topology $\tau_{\mathcal{S}}$: $p_{k,l}(f)=\sup_{x\in\mathbb{R}}|x|^k|f^{(l)}(x)|$, $k,l=0,1,2,\ldots$. Hence

$$\left|\Phi_n^{(+)}[f]\right| \leq M_n\sum_{k=0}^{n+2}\left(\begin{array}{c} n+2 \\ k \end{array}\right)p_{k,0}(f)<\infty,$$

so that $\Phi_n^{(+)}[f]$ is well defined for all $f(x)\in\mathcal{S}(\mathbb{R})$.

The fact that $\Phi_n^{(+)}$ is linear is clear: $\Phi_n^{(+)}[\alpha f+\beta g]=\alpha\Phi_n^{(+)}[f]+\beta\Phi_n^{(+)}[g]$, for all $f(x),g(x)\in\mathcal{S}(\mathbb{R})$ and $\alpha,\beta\in\mathbb{C}$.

To prove that $\Phi_n^{(+)}\in\mathcal{S}'(\mathbb{R})$ we need to prove that it is $\tau_{\mathcal{S}}$-continuous:

if $\tau_{\mathcal{S}}-\lim_{k,\infty}f_k(x)=f(x)$, then $\Phi_n^{(+)}[f_k]\to\Phi_n^{(+)}[f]$.

The inverted quantum oscillator: a distributional view

Step $\sharp\, 2$**:** Continuity of $\Phi_n^{(+)}$. A preliminary result.

Lemma:– *Given a sequence of functions* $\{f_k(x) \in \mathcal{S}(\mathbb{R})\}$, $\tau_{\mathcal{S}}$*-convergent to* $f(x) \in \mathcal{S}(\mathbb{R})$, *it follows that* $\{x^l f_k(x)\}$ *converges, in the norm* $\|.\|$ *of* $\mathcal{L}^2(\mathbb{R})$, *to* $x^l f(x)$, $\forall l \geq 0$.

The proof of this Lemma is based on similar estimates as those in **Step** $\sharp\, 1$.

Step $\sharp\, 3$**:** Back to the continuity of $\Phi_n^{(+)}$.

In view of **Step** $\sharp\, 2$ we have We have

$$\left|\Phi_n^{(+)}[f_k - f]\right| = \left|\langle\varphi_n^{(+)}, f_k - f\rangle\right| = \left|\left\langle \frac{\varphi_n^{(+)}}{(1+|x|)^{n+1}}, (1+|x|)^{n+1}(f_k - f) \right\rangle\right|,$$

with an obvious manipulation. Now, since for all n, k, $\frac{\varphi_n^{(+)}(x)}{(1+|x|)^{n+1}}, (1+|x|)^{n+1}(f_k(x) - f(x)) \in \mathcal{L}^2(\mathbb{R})$, the Schwarz inequality implies

$$\left|\Phi_n^{(+)}[f_k - f]\right| \leq \left\|\frac{\varphi_n^{(+)}}{(1+|x|)^{n+1}}\right\| \left\|(1+|x|)^{n+1}(f_k - f)\right\| \to 0$$

when $k \to \infty$, for all fixed $n \geq 0$. Hence $\Phi_n^{(+)}$ is continuous: $\Phi_n^{(+)} \in \mathcal{S}'(\mathbb{R})$.

The inverted quantum oscillator: a distributional view

Step $\sharp\, 4$: More results.

a. In the same way we can check that $\Phi_n^{(-)} \in \mathcal{S}'(\mathbb{R})$.

b. The relevance of distribution theory is also clear from the following result, counterpart of the previous one on $\Phi_n^{(\pm)}$:

For each fixed $n \geq 0$ the vector $\varphi_n^{(\pm)}(x)$ is a weak limit of $\varphi_n^{(\theta)}(x)$, for $\theta \to \pm\frac{\pi}{2}$:

$$\varphi_n^{(\pm)}(x) = w - \lim_{\theta, \pm\frac{\pi}{2}} \varphi_n^{(\theta)}(x),$$

i.e., $\forall f(x) \in \mathcal{S}(\mathbb{R})$,

$$\lim_{\theta, \pm\frac{\pi}{2}} \langle \varphi_n^{(\theta)}, f \rangle = \langle \varphi_n^{(\pm)}, f \rangle.$$

Analogously,

$$\psi_n^{(\pm)}(x) = w - \lim_{\theta, \pm\frac{\pi}{2}} \psi_n^{(\theta)}(x).$$

c. All these results show that the IQHO provides an interesting example of *weak pseudo-bosons*, i.e. of a suitable deformation of the CCR living *outside* $\mathcal{L}^2(\mathbb{R})$.

The inverted quantum oscillator: bi-coherent states

It is clear that we **cannot** use the theorem given for the Swanson-like Hamiltonian since the estimates

$$\|\varphi_n^{(\theta)}\| \leq K_\varphi\, r_\varphi^n M_n(\varphi), \qquad \|\psi_n^{(\theta)}\| \leq K_\psi\, r_\psi^n M_n(\psi).$$

make no sense here: $\|\varphi_n^{(\pm)}\| = \|\psi_n^{(\pm)}\| = \infty$.

Moreover, if we act as for the eigenstates of H (e.g., putting $\varphi_n^{(\pm)}(x) = \varphi_n^{(\pm\frac{\pi}{2})}(x))$), we get

$$\varphi^{(\pm)}(z;x) = \varphi^{(\pm\frac{\pi}{2})}(z;x) = \left(\frac{\Omega}{\pi}\right)^{1/4} \exp\left\{\pm i\,\frac{\pi}{8} - z_r^2 + \sqrt{2\Omega}\,e^{\pm i\pi/4}\,z\,x \mp \frac{i}{2}\,\Omega\,x^2\right\}$$

and

$$\psi^{(\pm)}(z;x) = \psi^{(\pm\frac{\pi}{2})}(z;x) = \left(\frac{\Omega}{\pi}\right)^{1/4} \exp\left\{\mp i\,\frac{\pi}{8} - z_r^2 + \sqrt{2\Omega}\,e^{\mp i\pi/4}\,z\,x \pm \frac{i}{2}\,\Omega\,x^2\right\} = \varphi^{(\mp)}(z;x).$$

Hence $|\varphi^{(\pm)}(z;x)|$ and $|\psi^{(\pm)}(z;x)|$ are **not** polynomials in $|x|$, contrarily to $|\varphi_n^{(\pm)}(x)|$ and $|\psi_n^{(\pm)}(x)|$and $\varphi^{(\pm)}(z;x)$ and $\psi^{(\pm)}(z;x)$ cannot give rise to tempered distributions. Still, they are not far from being elements of $\mathcal{S}'(\mathbb{R})$.

The inverted quantum oscillator: bi-coherent states

Construction of a (different) distributional framework

Let $\rho(x) > 0$ and

$$\mathcal{V}_\rho = \{f(x) \in \mathcal{L}^2(\mathbb{R}) : \rho(x)f(x) \in \mathcal{L}^2(\mathbb{R})\}.$$

If $\rho(x) \in \mathcal{L}^\infty(\mathbb{R})$, then $\mathcal{V}_\rho = \mathcal{L}^2(\mathbb{R})$.
Also, if $\rho(x)$ is continuous, but not necessarily bounded, $\mathcal{V}_\rho$ is dense in $\mathcal{L}^2(\mathbb{R})$, since $D(\mathbb{R}) \subset \mathcal{V}_\rho$. We can endow $\mathcal{V}_\rho$ with a topology, τ_ρ, as follows:

Definition:– *Given a sequence $\{f_n(x) \in \mathcal{V}_\rho\}$, we say that this is τ_ρ-convergent if (i) $\{f_n(x)\}$ is $\|.\|$-Cauchy and if (ii) $\{\rho(x)f_n(x)\}$ is $\|.\|$-Cauchy.*

The following Lemma holds:

Lemma:– *Suppose that $\rho^{-1}(x) \in \mathcal{L}^\infty(\mathbb{R})$. Then $\mathcal{V}_\rho$ is closed with respect to τ_ρ.*

Remark:– If $\{f_n(x) \in \mathcal{V}_\rho\}$ is τ_ρ-convergent to $f(x)$, $\{f_n(x) \in \mathcal{V}_\rho\}$ is also norm convergent to $f(x)$. The opposite implication does not hold except, for instance, if $\rho(x) \in \mathcal{L}^\infty(\mathbb{R})$. In this case if $\{f_n(x) \in \mathcal{V}_\rho\}$ is norm convergent to $f(x)$, then it also converges to $f(x)$ in τ_ρ.

The inverted quantum oscillator: bi-coherent states

Let $\rho(x)$ be as above, and let

$$\Theta_\rho = \{\Phi(x), \text{L-measurable: } \Phi(x)\rho^{-1}(x) \in \mathcal{L}^2(\mathbb{R})\}.$$

We have $\Theta_\rho \subseteq \mathcal{V}'_\rho$.
Indeed, since

$$F_\Phi[f] = \langle \Phi, f \rangle = \langle \Phi\rho^{-1}, f\rho \rangle,$$

and since, $\forall \Phi(x) \in \Theta_\rho$ and $f(x) \in \mathcal{V}_\rho$ we have $\Phi(x)\rho^{-1}(x), f(x)\rho(x) \in \mathcal{L}^2(\mathbb{R})$, $F_\Phi[f]$ is well defined and

$$|F_\Phi[f]| \leq \|\Phi\rho^{-1}\| \, \|f\rho\|.$$

Linearity of F_Φ is clear. F_Φ is also continuous: let $\{f_n(x) \in \mathcal{V}_\rho\}$ be a sequence τ_ρ convergent to $f(x) \in \mathcal{V}_\rho$. To show that $F_\Phi[f_k] \to F_\Phi[f]$ we simply observe that

$$|F_\Phi[f_k - f]| = \left|\langle \Phi\rho^{-1}, \rho(f_k - f)\rangle\right| \leq \|\Phi\rho^{-1}\|\|\rho(f_k - f)\| \to 0$$

for $k \to \infty$.

The inverted quantum oscillator: bi-coherent states

Back to the IQHO

We fix here $\rho(x) = e^{\Omega \frac{x^2}{2}}$, which is continuous. Hence $\mathcal{V}_\rho$ is dense in $\mathcal{L}^2(\mathbb{R})$, and it is τ_ρ-closed since $\rho^{-1}(x) = e^{-\Omega \frac{x^2}{2}} \in \mathcal{L}^\infty(\mathbb{R})$.

With this choice we see that $\varphi^{(\pm)}(z;x), \psi^{(\pm)}(z;x) \in \Theta_\rho$. Hence they belong to $\mathcal{V}'_\rho$. The resolution of the identity for these states is the following:

$$\int_{\mathbb{C}} \langle f, \psi^{(\pm)} \rangle \langle \varphi^{(\pm)}, g \rangle \frac{dz}{\pi} = \int_{\mathbb{C}} \langle f, \varphi^{(\pm)} \rangle \langle \psi^{(\pm)}, g \rangle \frac{dz}{\pi} = \langle f, g \rangle,$$

$\forall f(x), g(x) \in \mathcal{V}_\rho$. They are (weak) eigenstates of the annihilation operators. E.g.:

$$\langle f, A_\pm \varphi^{(\pm)}(z,x) \rangle = z \, \langle f, \varphi^{(\pm)}(z,x) \rangle,$$

$\forall f(x) \in \mathcal{V}_\rho$.

Then we conclude that:

1. $\varphi^{(\pm)}(z;x), \psi^{(\pm)}(z;x) \notin \mathcal{L}^2(\mathbb{R})$, $\varphi^{(\pm)}(z;x), \psi^{(\pm)}(z;x) \notin \mathcal{S}'(\mathbb{R})$, but they are continuous functionals on $\mathcal{V}_\rho$ (i.e., they are *sort of* distributions);
2. they satisfy some resolutions of the identity in $\mathcal{V}_\rho$;
3. they are (weak) eigenstates of the annihilation operators.

Concluding remarks

1. it is easy to slightly modify the Swanson model to get another deformed version of the Harmonic oscillator whose eigensystem can be deduced via pseudo-bosonic operators;
2. The natural functional space for this Swanson-like Hamiltonian, and for its bi-coherent states, is $\mathcal{L}^2(\mathbb{R})$;
3. The IQHO can be seen as a weak limit of the Swanson-like Hamiltonian. In particular...
4. ... the eigenstates of the IQHO are not in $\mathcal{L}^2(\mathbb{R})$, but they are tempered distributions;
5. ... they produce a resolution of the identity on $\mathcal{S}(\mathbb{R})$;
6. The bicoherent states are **not** in $\mathcal{L}^2(\mathbb{R})$, and **not even** in $\mathcal{S}'(\mathbb{R})$, but are *continuous functionals* on $\mathcal{V}_\rho$, an ad hoc space.

My main references *(so far)*

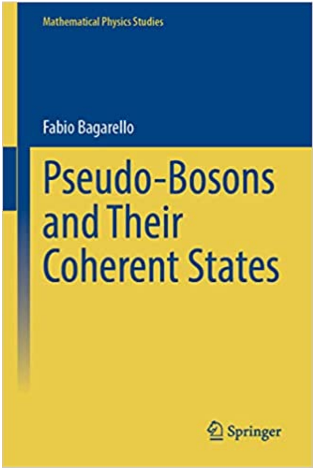

Figure: Springer, 2022