PHHQP seminar, October 20, 2022

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Noncommutative quantum mechanical systems associated with Lie algebras

[A.S., arXiv:2204.08705, J. Geom. Phys. **180** (2022) 104628]

NONCOMMUTATIVE GEOMETRY:

$[x_j, x_k] \neq 0.$

Physical interest:

- Noncommutative field theory: the fields $\phi(x)$, $A_j(x)$ defined on a noncommutative space.
  Not the subject of this talk.
- Noncommutative quantum mechanics: x_j treated as *dynamic variables* of a quantum system.

  [M. Chaichian, M. M. Sheikh-Jabbari and A. Tureanu, 2001; J. Gamboa, M. Loewe and J. R. Rojas, 2001]

MAIN STATEMENT:

Noncommutative QM is an ordinary QM in disguise.

BYPRODUCT:

Some nice formulas for the metrics of group manifolds and of S^n.

THE SYMPLEST NONCOM. SYSTEM:

$[x,y] \;=\; i\theta, \quad [\partial_j, x_k] = \delta_{jk}, \quad [\partial_j, \partial_k] = 0.$

(x, y are Hermitian)

• BASIC IDEA [F. Delduc, Q. Duret, F. Gieres and M. Lefrancois, 2008]

$$\begin{aligned} \hat{p}_j \;&= -i\partial_j \;\rightarrow\; X_j, \\ x \;&\rightarrow -i\frac{\partial}{\partial X} + \frac{\theta}{2}Y \;\equiv\; \hat{v}_X, \\ y \;&\rightarrow -i\frac{\partial}{\partial Y} - \frac{\theta}{2}X \;\equiv\; \hat{v}_Y. \end{aligned}$$

X, Y — new *commuting* coordinates, $\hat{v}_X, \hat{v}_Y$ — velocity operators.

• A naturally chosen Hamiltonian,

$$H = \frac{1}{2}(\hat{v}_X^2 + \hat{v}_Y^2),$$

describes the motion in the homogeneous magnetic field, $B = \theta$.

• A similar construction for

$$[x_a, x_b] \;=\; i\theta_{ab}, \quad [\partial_a, x_b] \;=\; \delta_{ab}.$$

Let $[x_a, x_b] = i\theta f_{abc} x_c$,

f_{abc} — structure constants of a Lie algebra.

•$[\partial_a, x_b] = \delta_{ab}$ is impossible, Jacobi identities not being fulfilled!

Remedy

[N. Durov, S. Meljanac, A. Samsarov, Z. Škoda, 2007]

• Postulate the commutators $[\tilde{\partial}_a, \tilde{\partial}_b] = 0$ and

$$[\tilde{\partial}_a, x_b] = \left[\frac{i\theta F(\tilde{\partial})}{e^{i\theta F(\tilde{\partial})} - 1}\right]_{ab}$$

$$= \delta_{ab} + \sum_{n=1}^{\infty} (-i\theta)^n \frac{B_n^+}{n!} [F^n(\tilde{\partial})]_{ab} \,,$$

Here $F_{ab}(V) = f_{acb} V_c$;

B_n^+ are the Bernoulli numbers, $B_0^+ = 1, B_1^+ = 1/2, B_2^+ = 1/6, B_4^+ = -1/30, \ldots$, $B_3^+ = B_5^+ = \ldots = 0$.

Then the Jacobi identities *are* fulfilled.

We represent

$$
\begin{aligned}
-i\tilde{\partial}_a &= X_a, \\
x_a = -i \left[\frac{F(\theta X)}{1 - e^{-F(\theta X)}}\right]_{ab} &\frac{\partial}{\partial X_b} \equiv -i\hat{D}_a \,.
\end{aligned}
$$

Theorem 1. *$\hat{D}_a$ coincide with the generators of right group rotations multiplied by θ and satisfying $[\hat{D}_a, \hat{D}_b] = -\theta f_{abc} \hat{D}_c$.*

Proof. We represent the group element as $\exp\{i\theta X_a t_a\}$. Then the shift of coordinates under an infinesimal right group rotation

$$
e^{i\theta X_a t_a} e^{i\theta \epsilon_a t_a} = e^{i\theta t_a (X_a + \delta X_a)}, \quad \epsilon \ll 1 \qquad (1)
$$

is

$$
\delta X_a = \sum_{n=0}^{\infty} \frac{B_n^+ [F^n(\theta X)]_{ab}}{n!} \epsilon_b + o(\epsilon) \,.
$$

• It follows from the general Baker-Campbell-Hausdorff-Dynkin formula.

The physical proof.

[R. Karplus and J. Schwinger, 1948; R. Feynman, 1951]

Let $\hat{H} = \hat{H}_0 + \hat{V}$, $\hat{V} \ll \hat{H}_0$. Then the *evolution operator*

$$U \;=\; e^{i(\hat{H}_0+\hat{V})t} \;=\; \lim_{N\to\infty}\left[1 + \frac{it}{N}(\hat{H}_0 + \hat{V})\right]^N .$$

is

$$U \;=\; e^{i\hat{H}_0 t} + i\int_0^t dt' e^{i(t-t')\hat{H}_0}\hat{V}e^{it'\hat{H}_0} + o(\hat{V}) .$$

Integrating, we derive

$$U \;=\; U_0\left(1 + i\hat{V}t + \frac{t^2}{2}[\hat{H}_0, \hat{V}] - \frac{it^3}{6}[\hat{H}_0, [\hat{H}_0, \hat{V}]] + \ldots\right),$$

Comparing this with (1), we can express ϵ_a via δX_a and then δX_a via ϵ_a. □

The natural *quantum* Hamiltonian is

$$\hat{H} \;=\; -\frac{1}{2}\hat{D}_a^2 . \tag{2}$$

The corresponding *classical* Hamiltonian reads

$$H^{\rm cl} \;=\; \frac{1}{2}g^{jk}P_j P_k , \tag{3}$$

where P_j are the canonical momenta and

$$g^{jk} \;=\; E_a{}^j E_a{}^k \tag{4}$$

with

$$E_a{}^j \;=\; \left[\frac{F(\theta X)}{1 - e^{-F(\theta X)}}\right]_a^{\;j} \,.$$

The classical Lagrangian corresponding to (3) is

$$L \;=\; \frac{1}{2} g_{jk} \dot{X}^j \dot{X}^k \,, \tag{5}$$

where $g_{jk} g^{kl} = \delta^l_k$. It describes the motion over the manifold with the metric g_{jk}. E^j_a and the inverse matrix E^a_j are the *vielbeins*.

Theorem 2. *The metric g_{jk} in (5) coincides with the invariant metric on the group manifold.*

Proof. The canonical metric on a group manifold, which is invariant under the left and right group rotations, is

$$g_{jk} \;=\; \frac{1}{h\theta^2} \mathrm{Tr}\left\{\partial_j \omega^{-1} \partial_k \omega\right\} \,,$$

where

$$\omega(x) \;=\; \exp\{i\theta X^j t_j\}$$

and t_j are the generators in a given representation with the Dynkin index h. Then $g_{jk}(X=0) = \delta_{jk}$.

Consider the distance between two close points $\omega(X+\delta X)$ and $\omega(X)$. For the invariant metric, this distance is the same as the distance between $\omega^{-1}(X)\omega(X+\delta X) = \omega(\epsilon)$ and the group unity. The latter is $ds^2 = \epsilon_a \epsilon_a$. To find the metric at the vicinity of X, we have only to express ϵ_a in terms of δX^j. To do so, we use the formula

$$\frac{1}{\theta}\,\omega^{-1}(-i\partial_j\omega) \;=\; \int_0^1 d\tau\, e^{-\tau R} t_j e^{\tau R}$$

$$=\; t_j - \frac{1}{2}[R, t_j] + \frac{1}{6}[R,[R,t_j]] - \ldots \;=\; E_j{}^a t_a$$

where $R = i\theta t_j X^j$ and

$$E_j{}^a \;=\; \left[\frac{1-e^{-F(\theta X)}}{F(\theta X)}\right]_j{}^a \;=\; \sum_{n=0}^{\infty} \frac{[F^n(-\theta X)]_j{}^a}{(n+1)!}$$

is the inverse vielbein. Then $\epsilon^a = E_j{}^a \delta X^j$ and the metric is

$$g_{jk} = E_j{}^a E_k{}^a\,, \tag{6}$$

which matches (4).

□

Theorem 3. *The Hamiltonian (2) coincides up to the factor $-1/2$ with the Laplace-Beltrami operator on the group manifold,*

$$\triangle \;=\; \frac{1}{\sqrt{g}}\partial_j(\sqrt{g}g^{jk})\partial_k\,,$$

with the invariant metric (6).

Proof. Clearly the operator $(\hat{D}_a)^2$ commutes with $\hat{D}_a$. This means that the Hamiltonian $\hat{H} = -\hat{D}_a^2/2$ is invariant under right group rotations. But it must also be invariant under left group rotations,

$$e^{i\theta t_j X^j} \;\to\; e^{i\theta t_j \epsilon^j} e^{i\theta t_j X^j}\,.$$

Indeed, right and left rotations commute and their generators $\hat{D}_a$ and $\hat{D}'_a$ must also commute. Both of them can be represented as infinite sums

$$\hat{D}_a \;=\; \sum_{n=0}^{\infty} \frac{B_n^+\,\theta^n[F^n(X)]_a{}^p}{n!}\partial_p\,,$$
$$\hat{D}'_a \;=\; \sum_{n=0}^{\infty} \frac{B_n^-\,\theta^n[F^n(X)]_a{}^p}{n!}\partial_p\,.$$

The only difference is that the sum for $\hat{D}_a$ involves the Bernoulli numbers B_n^+, while the sum for $\hat{D}'_a$

involves the Bernoulli numbers B_n^-. The latter coincide with B_n^+ for all n except $n = 1$ where $B_1^- = -B_1^+ = -1/2$.

Left and right rotations commute as well as their generators. The vanishing of the commutator $[\hat{D}'_a, \hat{D}_b] = 0$ entails the vanishing of $[\hat{D}'_a, \hat{H}^{\mathrm{qu}}]$. The only second order differential operator that is invariant under both left and right group rotations is the Laplace-Beltrami operator.

□

The $SU(2)$ case.

• For $SU(2)$, the series for the vielbeins and the metric can be resummed:

$$
\begin{aligned}
E_a{}^j &= \delta_{aj} + \frac{\theta}{2}\varepsilon_{apj}X_p + \\
&\quad (X_aX_j - X^2\delta_{ja})\left[\frac{1}{X^2} - \frac{\theta}{2X}\cot\frac{\theta X}{2}\right], \\
E_j{}^a &= \delta_{ja} - 2\frac{\sin^2\frac{\theta X}{2}}{\theta X^2}\varepsilon_{jpa}X_p + \\
&\quad \frac{1}{X^2}\left[1 - \frac{\sin(\theta X)}{\theta X}\right](X_jX_a - X^2\delta_{ja}),
\end{aligned}
$$

$$
\begin{aligned}
g^{jk} &= A^{-1}(X)\left(\delta_{jk} - \frac{X_jX_k}{X^2}\right) + \frac{X_jX_k}{X^2}, \\
g_{jk} &= A(X)\left(\delta_{jk} - \frac{X_jX_k}{X^2}\right) + \frac{X_jX_k}{X^2}, \qquad (7)
\end{aligned}
$$

where $X = \sqrt{X_pX_p}$ and

$$A(X) = 4\frac{\sin^2\frac{\theta X}{2}}{\theta^2X^2}.$$

• The same formulas (but without the metric interpretation) were derived in recent [V.G. Kupriyanov, 2021].

The corresponding scalar curvature is $R = 3\theta^2/2$. Bearing in mind that $R = 6/\rho^2$ where ρ is the radius of S^3, we derive

$$\rho \; = \; \frac{2}{\theta}\,.$$

Alternative representation

• For $su(2)$, the algebra $[x_a, x_b] = i\theta\epsilon_{abc}x_c \; \longrightarrow$ $[\hat{D}_a, \hat{D}_b] = -\theta\epsilon_{abc}\hat{D}_c$ allows for a simple representation,

$$\hat{D}_a \; = \; \partial_a - \frac{\theta}{2}\varepsilon_{abc}Y_b\,\partial_c + \frac{\theta^2}{4}Y_aY_b\,\partial_b\,.$$

(It was derived with a different interpretation in [G. Gubitosi, F. Lizzi, J.J. Relancio and P. Vitale, 2021]).

This gives the following metric,

$$\begin{aligned} g^{jk} \; &= \; \delta^{jk} + \kappa(Y^2\delta^{jk} + Y^jY^k) + \kappa^2Y^2Y^jY^k\,, \\ g_{jk} \; &= \; \frac{\delta_{jk}}{1+\kappa Y^2} - \frac{\kappa Y_jY_k}{(1+\kappa Y^2)^2}\,, \end{aligned} \qquad (8)$$

where $\kappa = \theta^2/4$.

The derived metrics are interrelated and related to the familiar conformally flat metric on S^3,

$$g_{jk} \ = \ \frac{\delta_{jk}}{[1 + Z^2/(4\rho^2)]^2} \,,$$

by the following changes of coordinates:

$$Y^p \ = \ X^p \, \frac{2 \tan(\theta X/2)}{\theta X} \ = \ Z^p \, \frac{1}{1 - Z^2/(4\rho^2)} \,. \quad (9)$$

Note that the value $Y = \infty$ describes the *equator* of S^3. In other words, the metric (8) describes only a half of $SU(2) \equiv S^3$, which is $SO(3) \equiv RP^3$.

OTHER SPHERES

The coordinate changes (9) work for any sphere S^n, hence the expressions (7), (8) describe the metric of any S^n.

Two specific cases

1. $S^3 \equiv SU(2)$ is related to $su(2)$ Lie algebra.

2. S^7 is related to the following *nonlinear* algebra:

$$[\hat{D}_A, X_B] = \delta_{AB} - \frac{\theta}{2}\eta_{ABC}X_C +$$
$$(X_A X_B - \delta_{AB}X^2)\left[\frac{1}{X^2} - \frac{\theta}{2X}\cot\frac{\theta X}{2}\right],$$
$$[\hat{D}_A, \hat{D}_B] = -\theta\,\eta_{ABC}\hat{D}_C - 4\theta\frac{\sin\theta X}{X}\eta_{ABCD}X_C\hat{D}_D$$
$$+\, 8\frac{\sin^2\frac{\theta X}{2}}{X^2}\eta_{ABCD}\,\eta_{DEF}X_C X_E\hat{D}_F,$$

where $A, B, \ldots = 1, \ldots, 7$, η_{ABC} is the octonion multiplication tensor, $e_A e_B = \eta_{ABC}e_C$, and $\eta_{ABCD} = \epsilon_{ABCDEFG}\,\eta_{EFG}$.

[V.G. Kupriyanov, 2018]

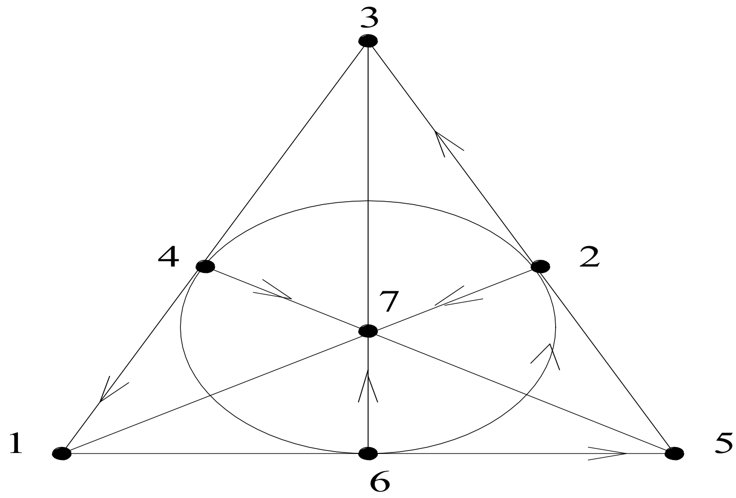

Figure 1: Fano graph.

GUREVICH-SAPONOV $U(N)$ MODEL [2015, 2020]

Algebra

$$
\begin{aligned}
[l_i^m, l_k^n] &= \theta(\delta_i^n l_k^m - \delta_k^m l_i^n), \\
[\tilde{\partial}_i^m, l_k^n] &= \delta_i^n \delta_k^m + \theta \delta_i^n \tilde{\partial}_k^m ,
\end{aligned}
$$

- Jacobi indentities satisfied.

Variable change

$$
x_A \;=\; (t_A)_m^i \, l_i^m, \qquad \tilde{\partial}_A \;=\; 2(t_A)_m^i \, \tilde{\partial}_i^m \,.
$$

Here $A = (0, a)$, $x_A = (-i\tau, x_a)$, $\tilde{\partial}_A = (i\tilde{\partial}_\tau, \tilde{\partial}_a)$, t_A are $U(N)$ generators, $\mathrm{Tr}\{t_A t_B\} = \delta_{AB}/2$.

In particular:

$$
\tau = \frac{i}{\sqrt{2N}}\, l_j^j, \qquad \tilde{\partial}_\tau \;=\; -i\sqrt{\frac{2}{N}}\, \tilde{\partial}_j^j, \qquad t_0 \;=\; \frac{1}{\sqrt{2N}}\, \mathbb{1} \,.
$$

$N = 2$

$$
\begin{aligned}
[x_a, x_b] &= i\theta\epsilon_{abc}x_c, \qquad [x_a, \tau] = 0, \\
[\tilde{\partial}_a, x_b] &= \delta_{ab}\left(1 + \frac{i\theta}{2}\tilde{\partial}_\tau\right) + \frac{i\theta}{2}\epsilon_{abc}\tilde{\partial}_c, \\
[\tilde{\partial}_\tau, x_a] &= -i\frac{\theta}{2}\tilde{\partial}_a, \quad [\tilde{\partial}_a, \tau] = i\frac{\theta}{2}\tilde{\partial}_a, \\
[\tilde{\partial}_\tau, \tau] &= 1 + \frac{i\theta}{2}\tilde{\partial}_\tau \,.
\end{aligned}
$$

- No infinite series in derivatives!
- x_a, τ are Hermitian and $\tilde{\partial}_a, \tilde{\partial}_\tau$ are anti-Hermitian.

Representation

$$
\begin{aligned}
&\tilde{\partial}_a \to X_a, \quad \tilde{\partial}_\tau \to T, \\
&x_a \to -i\hat{D}_a = \\
&-i\left[\left(1 + \frac{\theta T}{2}\right)\frac{\partial}{\partial X_a} - \frac{\theta}{2}X_a\frac{\partial}{\partial T} + \frac{\theta}{2}\epsilon_{abc}X_b\frac{\partial}{\partial X_c}\right], \\
&\tau \to -i\hat{D}_0 = \\
&-i\left[\left(1 + \frac{\theta T}{2}\right)\frac{\partial}{\partial T} + \frac{\theta}{2}X_a\frac{\partial}{\partial X_a} + \theta\right].
\end{aligned}
$$

- θ added in the R.H.S. of the second line to make $-i\hat{D}_0$ Hermitian.

- The algebra

$$[\hat{D}_a, \hat{D}_b] = -\theta\epsilon_{abc}\hat{D}_c, \quad [\hat{D}_a, \hat{D}_0] = 0$$

holds.

Consider the quantum Hamiltonian

$$\hat{H}^{\rm qu} \;=\; -\frac{1}{2}(\hat{D}_0\hat{D}_0 + \hat{D}_a\hat{D}_a)\,.$$

with the classical counterpart

$$H^{\rm cl} \;=\; \frac{P_0^2 + P_a^2}{2}\left[1 + \theta T + \frac{\theta^2}{4}(T^2 + X_a^2)\right]\,.$$

This Hamiltonian describes the motion of a particle over a 4-dimensional manifold with the conformally flat metric

$$g_{\mu\nu} = \;\frac{\delta_{\mu\nu}}{F(T, X_a)} \;=\; \frac{\delta_{\mu\nu}}{1 + \theta T + \theta^2(T^2 + X_a^2)/4}\,. \quad (10)$$

The Ricci tensor has the following components:

$$R_{00} \;=\; \frac{\theta^4}{8F^2}, \quad R_{0a} \;=\; -\frac{\theta^3(1 + \theta T/2)X_a}{4F^2},$$
$$R_{ab} \;=\; \frac{\theta^2}{2F}\left(\delta_{ab} - \frac{\theta^2 X_a X_b}{4F}\right)\,.$$

An explicit calculation shows that all the curvature invariants made from the Ricci tensor are constants:

$$R = R_{\mu\nu}g^{\mu\nu} \;=\; \frac{3\theta^2}{2}\,,$$
$$R_{\mu\nu}R^{\mu\nu} \;=\; \frac{3\theta^4}{4}, \quad R_{\mu\nu}R^{\nu\rho}R_\rho{}^\mu \;=\; \frac{3\theta^6}{8}, \quad \text{etc.}$$

They are the same as for a *3-dimensional* “round” sphere of radius $\rho = 2/\theta$.

- Eq. (10) gives a nontrivial metric on $U(2)$!

Higher N

The algebra:

$$
\begin{aligned}
[x_a, x_b] &= i\theta f_{abc} x_c, \qquad [x_a, \tau] = 0, \\
[\tilde{\partial}_a, x_b] &= \delta_{ab}\left(1 + \frac{i\theta}{\sqrt{2N}}\tilde{\partial}_\tau\right) + \frac{\theta}{2}(i f_{abc} + d_{abc})\tilde{\partial}_c, \\
[\tilde{\partial}_\tau, x_a] &= -i\frac{\theta}{\sqrt{2N}}\tilde{\partial}_a, \quad [\tilde{\partial}_a, \tau] = i\frac{\theta}{\sqrt{2N}}\tilde{\partial}_a, \\
[\tilde{\partial}_\tau, \tau] &= 1 + \frac{i\theta}{\sqrt{2N}}\tilde{\partial}_\tau .
\end{aligned}
$$

- No possibility to make the momenta $\hat{P}_a = -i\hat{D}_a$, $\hat{P}_0 = -i\hat{D}_0$ Hermitian.
- No possibility to define a Hermitian Hamiltonian.
- No possibility to determine the metric of $U(N)$.