Ladder operators in compatible spaces and the BCH formula

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Organization of the talk

Part 1: ladder operators in compatible spaces

- *a short list of existing models (in $\mathcal{H}$)*
- *our operators...*
- *...and their compatible families of vectors*
- *Non self-adjoint Hamiltonians connected to these, as (generalized) eigenfamilies*
- *Weak bi-coherent states*

Part 2: the Baker-Campbell-Hausdorff (BCH) formula

- *Motivations and a comparison with the Weyl group*
- *Displacement operator: a **lot** of estimates!*
- *Generalizations for non self-adjoint operators: displacement-like operators*

An historical note

In the literature, along the years, Hamiltonians like these have been considered:

$$H_1 = \frac{\nu}{2}\left(p_1^2 + x_1^2 + p_2^2 + x_2^2\right) + i\sqrt{2}\,(p_1 + p_2),$$

$$H_2 = \frac{1}{2}(p_1^2 + x_1^2) + \frac{1}{2}(p_2^2 + x_2^2) + i\left[A(x_1 + x_2) + B(p_1 + p_2)\right],$$

$$H_3 = \frac{1}{2}\left(p^2 + x^2\right) - \frac{i}{2}\,\tan(2\theta)\left(p^2 - x^2\right),$$

$$H_4 = \frac{1}{2}\left(p^2 + x^2\right) + \frac{i}{\sqrt{2}}(\beta - \alpha)\,p + \frac{1}{\sqrt{2}}(\beta + \alpha)\,x,$$

$$H_5 = (p_1^2 + x_1^2) + (p_2^2 + x_2^2 + 2ix_2) + 2\varepsilon x_1 x_2,$$

and many others. A common feature is that they can all be written in terms of operators A_j and $B_j (\neq A_j^\dagger)$, $j = 1, 2$, which behave as ladder operators. We find $H_j = B_1 A_1 + B_2 A_2$, or similar.

An historical note

In particular (in $d = 1$), these ladder operators are of the following forms:

Harmonic oscillator:

$$a = c = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + x\right), \qquad b = c^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + x\right),$$

shifted (in space) harmonic oscillator:

$$a = c + \alpha 1\!\!1 = \frac{1}{\sqrt{2}}\frac{d}{dx} + \frac{1}{\sqrt{2}}(x + \sqrt{2}\alpha 1\!\!1) \qquad b = c^\dagger + \beta 1\!\!1 = -\frac{1}{\sqrt{2}}\frac{d}{dx} + \frac{1}{\sqrt{2}}(x + \sqrt{2}\beta 1\!\!1),$$

for some complex α and β with $\alpha \neq \overline{\beta}$.

Swanson model:

$$a = \frac{1}{\sqrt{2}}\left(e^{-i\theta}\frac{d}{dx} + e^{i\theta}x\right), \qquad b = \frac{1}{\sqrt{2}}\left(-e^{-i\theta}\frac{d}{dx} + e^{i\theta}x\right).$$

Pseudo-bosonic superpotentials (PBSs):

$$a = \frac{d}{dx} + w_a(x), \qquad b = -\frac{d}{dx} + w_b(x),$$

with $(w_a(x) + w_b(x))' = 1$.

An historical note...and our program

They are all of the form

$$a=\alpha_a(x)\,\frac{d}{dx}+\beta_a(x),\qquad b=-\frac{d}{dx}\,\alpha_b(x)+\beta_b(x),$$

for suitable $\beta_a(x)$ and $\beta_b(x)$, and constant $\alpha_a(x)$ and $\alpha_b(x)$.

Our interest, here, is to consider the following questions:

1. do these operators obey pseudo-bosonic commutation rules?
2. do they produce biorthonormal families of vectors?
3. do these vectors belong to $\mathcal{L}^2(\mathbb{R})$? Or, in case they do not, are these families compatible (its meaning later)?
4. are these vectors (generalized) eigenvectors of some particular operator?
5. are a and b connected to some families of bi-coherent states?
6. do these bi-coherent states produce some sort of resolution of the identity?

The first steps...

The first question is easily answered: in order for $[a,b]=1\!\!1$, first of all $[a,b]f(x)$ must make sense, and then $[a,b]f(x)=f(x)$. This is true if $\alpha_j(x)$ and $\beta_j(x)$, $j=a,b$, are *regular* (e.g., $\alpha_j(x),\beta_j(x)\in C^\infty$) and satisfy the following equalities

$$\begin{cases} \alpha_a(x)\alpha_b'(x)=\alpha_a'(x)\alpha_b(x), \\ \alpha_a(x)\beta_b'(x)+\alpha_b(x)\beta_a'(x)=1+\alpha_a(x)\alpha_b''(x). \end{cases}$$

These equations are satisfied in all cases listed above ($\alpha_j(x)$ constant, $j=a,b$), and for the PBSs ($\alpha_j(x)=1$) if $(\beta_a(x)+\beta_b(x))'=1$.
Let us introduce the adjoint of a and b:

$$a^\dagger=-\frac{d}{dx}\,\overline{\alpha_a(x)}+\overline{\beta_a(x)}, \qquad b^\dagger=\overline{\alpha_b(x)}\,\frac{d}{dx}+\overline{\beta_b(x)},$$

and let us proceed us for ordinary PBs: the vacua of a and $b^\dagger$, $a\varphi_0(x)=0$ and $b^\dagger\psi_0(x)=0$, are:

$$\varphi_0(x)=N_\varphi\exp\left\{-\int\frac{\beta_a(x)}{\alpha_a(x)}\,dx\right\}, \qquad \psi_0(x)=N_\psi\exp\left\{-\int\frac{\overline{\beta_b(x)}}{\overline{\alpha_b(x)}}\,dx\right\},$$

which are well defined under our assumptions if $\alpha_j(x)\neq 0\ \forall x\in\mathbb{R}$. Of course, we don't know if $\varphi_0(x),\psi_0(x)\in\mathcal{L}^2(\mathbb{R})$ or not. However...

The first steps...

If we now introduce

$$\varphi_n(x) = \frac{1}{\sqrt{n!}}\, b^n \varphi_0(x), \qquad \psi_n(x) = \frac{1}{\sqrt{n!}}\, (a^\dagger)^n \psi_0(x),$$

$n \geq 0$, we can prove the following:

Proposition:– Calling $\theta(x) = \alpha_a(x)\beta_b(x) + \alpha_b(x)\beta_a(x)$ we have

$$\varphi_n(x) = \frac{1}{\sqrt{n!}}\, \pi_n(x)\varphi_0(x), \qquad \psi_n(x) = \frac{1}{\sqrt{n!}}\, \sigma_n(x)\psi_0(x),$$

$n \geq 0$, where $\pi_n(x)$ and $\sigma_n(x)$ are defined recursively as follows:

$$\pi_0(x) = 1, \qquad \pi_n(x) = \left(\frac{\theta(x)}{\alpha_a(x)} - \alpha_b'(x)\right)\pi_{n-1}(x) - \alpha_b(x)\pi_{n-1}'(x),$$

$$\sigma_0(x) = 1, \qquad \sigma_n(x) = \overline{\left(\frac{\theta(x)}{\alpha_b(x)} - \alpha_a'(x)\right)}\,\sigma_{n-1}(x) - \overline{\alpha_a(x)}\,\sigma_{n-1}'(x),$$

$n \geq 1$.

Question: is it possible to write $\pi_n(x)$ and $\sigma_n(x)$ is a simpler way? **YES!**

Back to constant $\alpha_j(x)$, for a moment

If $\alpha_a(x)=\alpha_a$ and $\alpha_b(x)=\alpha_b$, both non zero, $\theta(x)=\alpha_a\beta_b(x)+\alpha_b\beta_a(x)=x+k$, $k\in\mathbb{C}$, so that $\pi_0(x)=\sigma_0(x)=1$ and

$$\pi_n(x)=\frac{1}{\alpha_a}(x+k)\,\pi_{n-1}(x)-\alpha_b\,\pi'_{n-1}(x),\qquad \sigma_n(x)=\frac{1}{\overline{\alpha}_a}(x+\overline{k})\,\sigma_{n-1}(x)-\overline{\alpha}_a\,\sigma'_{n-1}(x).$$

We can then prove that

$$\pi_n(x)=\sqrt{\left(\frac{\alpha_b}{2\alpha_a}\right)^n}H_n\left(\frac{x+k}{\sqrt{2\alpha_a\alpha_b}}\right),\qquad \sigma_n(x)=\sqrt{\left(\frac{\overline{\alpha}_b}{2\overline{\alpha}_a}\right)^n}H_n\left(\frac{x+\overline{k}}{\sqrt{2\overline{\alpha}_a\overline{\alpha}_b}}\right).$$

We also get

$$\varphi_0(x)=N_\varphi\exp\left\{-\frac{1}{\alpha_a}\int\beta_a(x)\,dx\right\},\qquad \psi_0(x)=N_\psi\exp\left\{-\frac{1}{\overline{\alpha_b}}\int\overline{\beta_b(x)}\,dx\right\},$$

where $\alpha_a\beta_b(x)+\alpha_b\beta_a(x)=x+k=\theta(x)$.
We have $\varphi_n(x)\,\overline{\Psi_m(x)}\in\mathcal{L}^1(\mathbb{R})$ (i.e., $\varphi_n(x)$ and $\Psi_m(x)$ are **compatible**), for all $n,m\geq 0$. Indeed $\varphi_n(x)\,\overline{\Psi_m(x)}$ is nothing but the a polynomial of degree $n+m$ times the exponential

$$e^{-\int\left(\frac{\beta_a(x)}{\alpha_a}+\frac{\beta_b(x)}{\alpha_b}\right)dx}=e^{-\frac{1}{\alpha_a\alpha_b}\int\theta(x)\,dx}=e^{-\frac{1}{\alpha_a\alpha_b}\left(\frac{x^2}{2}+kx+\tilde{k}\right)},$$

for some integration constant $\tilde{k}$, which goes to zero very fast if $\alpha_a\alpha_b>0$.

A more general choice: x-dependent $\alpha_j(x)$

We take now $\alpha_a(x) = \alpha_b(x) = \alpha(x)$, where $\alpha(x) \neq 0$ for all $x \in \mathbb{R}$. In this case the first equality for $\alpha_j(x)$ and $\beta_j(x)$ is automatically true, $\forall \alpha(x)$. The second equation becomes

$$(\beta_a(x) + \beta_b(x))' = \frac{1}{\alpha(x)} + \alpha''(x),$$

which is solved (for instance) by $\beta_a(x)$ and $\beta_b(x)$ as follows:

$$\beta_a(x) = \int \frac{dx}{\alpha(x)}, \qquad \beta_b(x) = \alpha'(x).$$

We need $\beta_a(x)$ (or $\beta_b(x)$) to be an increasing function (true, e.g., if $\alpha(x) > 0$). Then

$$\pi_n(x) = \beta_a(x)\, \pi_{n-1}(x) - \frac{1}{\beta_a'(x)} \pi_{n-1}'(x),$$

with $\pi_0(x) = 1$, $n \geq 1$, which produces

$$\pi_n(x) = \frac{1}{\sqrt{2^n}}\, H_n\left(\frac{\beta_a(x)}{\sqrt{2}}\right),$$

for all $n \geq 0$.

Claim:– $H_n(.)$ *always appear because the commutation rule* $[a, b] = 1\!\!1$!

As for the functions $\varphi_n(x)$ and $\psi_n(x)$....

A more general choice: x-dependent $\alpha_j(x)$

... the vacua are

$$\varphi_0(x) = N_\varphi \exp\left\{-\frac{1}{2}(\beta_a(x))^2\right\}, \qquad \psi_0(x) = \frac{N_\psi}{\alpha(x)},$$

if $\alpha(x)$ is real. Putting all together we conclude that

$$\varphi_n(x) = \frac{N_\varphi}{\sqrt{2^n n!}} H_n\left(\frac{\beta_a(x)}{\sqrt{2}}\right) e^{-\left(\frac{\beta_a(x)}{\sqrt{2}}\right)^2}, \quad \psi_n(x) = \frac{N_\psi}{\sqrt{2^n n!}} H_n\left(\frac{\beta_a(x)}{\sqrt{2}}\right) \frac{1}{\alpha(x)}.$$

It is clear that, in general, $\varphi_n(x)$ and $\psi_n(x)$ need not being square-integrable. However....under very mild assumption on $\alpha(x)$, the families $\mathcal{F}_\varphi = \{\varphi_n(x)\}$ and $\mathcal{F}_\psi = \{\psi_n(x)\}$ are compatible and biorthonormal (in our slightly extended sense.) Indeed we have

$$\langle\psi_m, \varphi_n\rangle = \frac{\overline{N}_\psi N_\varphi}{\sqrt{2^{n+m}\, n!\, m!}} \int_{-\infty}^{\infty} H_m\left(\frac{\beta_a(x)}{\sqrt{2}}\right) H_n\left(\frac{\beta_a(x)}{\sqrt{2}}\right) e^{-\left(\frac{\beta_a(x)}{\sqrt{2}}\right)^2} \frac{dx}{\alpha(x)}.$$

Introducing $s = \frac{\beta_a(x)}{\sqrt{2}}$ we get

$$\langle\psi_m, \varphi_n\rangle = \frac{\overline{N}_\psi N_\varphi}{\sqrt{2^{n+m-1}\, n!\, m!}} \int_{-\infty}^{\infty} H_m(s) H_n(s) e^{-s^2} ds = \delta_{n,m},$$

if $\overline{N}_\psi N_\varphi = \frac{1}{\sqrt{2\pi}}$.

A more general choice: x-dependent $\alpha_j(x)$

What about resolution of the identity?

It is clear that this cannot be true on all of $\mathcal{L}^2(\mathbb{R})$. In fact, if $\alpha(x)$ is such that $\psi_n(x) \notin \mathcal{L}^2(\mathbb{R})$, then it is necessary to work with a dual space of *regular functions*, as in distribution theory.
With this in mind, let us introduce the set

$$\mathcal{E} = \left\{ h(s) \in \mathcal{L}^2(\mathbb{R}) : \, h_-(s) := h(\beta_a^{-1}(\sqrt{2}s))\, e^{s^2/2} \in \mathcal{L}^2(\mathbb{R}) \right\}$$

This set is dense in $\mathcal{L}^2(\mathbb{R})$, since it contains the set $\mathcal{D}(\mathbb{R})$ of all the compactly supported C^∞ functions. In fact, it is easy to see that $h_-(s)$ is compactly supported and continuous. Hence the integral of its square modulus exists. In particular, if β_a^{-1} is C^∞, then $h_-(s) \in \mathcal{D}(\mathbb{R})$ for all $h(x) \in \mathcal{D}(\mathbb{R})$.

Moreover, if $h(x) \in \mathcal{E}$, then the function

$$h_+(s) := h(\beta_a^{-1}(\sqrt{2}s))\, \alpha(\beta_a^{-1}(\sqrt{2}s))\, e^{-s^2/2} \in \mathcal{L}^2(\mathbb{R})$$

as well, at least under very general conditions on $\alpha(x)$. This is because $|h_+(s)|^2 = |h_-(s)|^2 |g(s)|^2$, where $g(s) = \alpha(\beta_a^{-1}(\sqrt{2}s))\, e^{-s^2}$. Now, it is sufficient that $g(s) \in \mathcal{L}^\infty(\mathbb{R})$ to conclude that $h_+(s) \in \mathcal{L}^2(\mathbb{R})$. (However, even if $\alpha(x)$ diverges very fast, if $h(x) \in \mathcal{D}(\mathbb{R})$ then $h_+(s) \in \mathcal{L}^2(\mathbb{R})$ anyhow, which is what we will use in the following.)

A more general choice: x-dependent $\alpha_j(x)$

Theorem:– $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ are $\mathcal{E}$-quasi bases.

This is based on the equalities

$$\langle f, \varphi_n\rangle = N_\varphi\, \pi^{1/4}\sqrt{2}\langle f_+, e_n\rangle, \qquad \langle \psi_n, g\rangle = \overline{N}_\psi\, \pi^{1/4}\sqrt{2}\langle e_n, g_-\rangle.$$

Here $e_n(s) = \frac{1}{\sqrt{2^n n!\sqrt{\pi}}} H_n(s) e^{-s^2/2}$ is the n-th eigenstate of the quantum harmonic oscillator.

Then:

$(f(x), \varphi_n(x))$ and $(g(x), \psi_n(x))$ are compatible, $\forall n \geq 0$.

Moreover, from these equalities we deduce that

$$\sum_{n=0}^{\infty}\langle f, \varphi_n\rangle\langle \psi_n, g\rangle = \sum_{n=0}^{\infty}\langle f, \psi_n\rangle\langle \varphi_n, g\rangle = \langle f, g\rangle,$$

$\forall f, g \in \mathcal{E}$.

Example 1:

The simplest situation is when $\alpha_a(x) = \alpha_b(x) = 1$. Hence $\beta_a(x) = x$ (increasing, as required) and $\beta_b(x) = k$, $k \in \mathbb{R}$. We have already found

$$\varphi_n(x) = \frac{N_\varphi}{\sqrt{2^n n!}} H_n\left(\frac{x+k}{\sqrt{2}}\right) e^{-x^2/2} \in \mathcal{L}^2(\mathbb{R}),$$

and

$$\psi_n(x) = \frac{N_\psi}{\sqrt{2^n n!}} H_n\left(\frac{x+k}{\sqrt{2}}\right) e^{-kx} \notin \mathcal{L}^2(\mathbb{R}).$$

However $\varphi_n(x)\psi_m(x) \in \mathcal{L}^1(\mathbb{R})$, $\forall n, m \geq 0$.

These are *generalized eigenstates* of H and $H^\dagger$ with eigenvalues $0, 1, 2, 3, \ldots$, where

$$H = ba = -\frac{d^2}{dx^2} + (k-x)\frac{d}{dx} + (kx-1),$$

and

$$H^\dagger = a^\dagger b^\dagger = -\frac{d^2}{dx^2} + (x-k)\frac{d}{dx} + kx.$$

Example 2:

We take $\alpha(x) = \frac{1}{1+x^2}$. Hence $\beta_a(x) = x + \frac{x^3}{3}$ and $\beta_b(x) = \frac{-2x}{(1+x^2)^2}$. We find

$$\varphi_0(x) = N_\varphi \exp\left\{-\frac{1}{2}(x + x^3/3)^2\right\}, \qquad \psi_0(x) = N_\psi\,(1 + x^2).$$

It is clear that $\varphi_0(x) \in \mathcal{L}^2(\mathbb{R})$, while $\psi_0(x)$ is not square-integrable. Furthermore,

$$\pi_n(x) = \left(x + \frac{x^3}{3}\right)\pi_{n-1}(x) - \frac{1}{(1+x^2)}\,\pi'_{n-1}(x),$$

with $\pi_0(x) = 1$, with a similar expression for $\sigma_n(x)$. More explicitly we get

$$\pi_n(x) = \sigma_n(x) = \frac{1}{\sqrt{2^n}} H_n\left(\frac{x + x^3/3}{\sqrt{2}}\right),$$

so that

$$\varphi_n(x) = \frac{N_\varphi}{\sqrt{2^n n!}}\,H_n\left(\frac{x + x^3/3}{\sqrt{2}}\right) e^{-\frac{1}{2}(x+x^3/3)^2} \in \mathcal{L}^2(\mathbb{R}),$$

and

$$\psi_n(x) = \frac{N_\psi}{\sqrt{2^n n!}}\,H_n\left(\frac{x + x^3/3}{\sqrt{2}}\right)(1 + x^2) \notin \mathcal{L}^2(\mathbb{R}),$$

$n \geq 0$. If

$$\overline{N}_\psi N_\varphi = \frac{1}{\sqrt{2\pi}} \qquad \Rightarrow \qquad \langle\psi_m, \varphi_n\rangle = \delta_{n,m},$$

$\forall n, m \geq 0$.

Example 2:

We can conclude that $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ are **compatible** biorthonormal $\mathcal{E}$-quasi bases.

Moreover, $\varphi_n(x)$ and $\psi_n(x)$ are *generalized eigenstates*, respectively, of

$$H = ba = -\frac{1}{(1+x^2)^2}\frac{d^2}{dx^2} - \frac{x(-3+7x^2+5x^4+x^6)}{3(1+x^2)^3}\frac{d}{dx} - 1,$$

and

$$H^\dagger = a^\dagger b^\dagger =$$

$$= -\frac{1}{(1+x^2)^2}\frac{d^2}{dx^2} + \frac{x(21+7x^2+5x^4+x^6)}{3(1+x^2)^3}\frac{d}{dx} - \frac{2(-3+18x^2+7x^4+5x^6+x^8)}{3(1+x^2)^4},$$

with eigenvalues $\mathbb{N}\cup\{0\}$.

Example 3:

Let $\alpha_a(x) = 2\alpha_b(x) = \frac{1}{\cosh(x)}$. Then $\beta_a(x) = \int \frac{dx}{\alpha_b(x)} = 2\sinh(x)$ and $\beta_b(x) = \alpha_b'(x) = \frac{-\sinh(x)}{2(\cosh(x))^2}$. The vacua are

$$\varphi_0(x) = N_\varphi \exp\left\{-(\cosh(x))^2\right\}, \qquad \psi_0(x) = 2N_\psi\,\cosh(x).$$

Clearly $\varphi_0(x) \in \mathcal{L}^2(\mathbb{R})$, while $\psi_0(x) \notin \mathcal{L}^2(\mathbb{R})$. However, $\overline{\psi_0(x)}\,\varphi_0(x) \in \mathcal{L}^1(\mathbb{R})$. Indeed, for $|x| \to \infty$, $\psi_0(x) \simeq e^{|x|}$, but $\varphi_0(x) \simeq e^{-e^{|x|}}$. Hence they are compatible. Further we have $\pi_0(x) = \sigma_0(x) = 1$, and

$$\pi_n(x) = \sinh(x)\pi_{n-1}(x) - \frac{1}{2\cosh(x)}\,\pi_{n-1}'(x) = \frac{1}{2^n}\,H_n(\sinh(x)),$$

while

$$\sigma_n(x) = 2\sinh(x)\sigma_{n-1}(x) - \frac{1}{\cosh(x)}\,\sigma_{n-1}'(x) = H_n(\sinh(x)).$$

$\forall n \geq 0$, so that

$$\varphi_n(x) = \frac{N_\varphi}{2^n\sqrt{n!}}\,H_n(\sinh(x))e^{-(\cosh(x))^2}, \quad \psi_n(x) = \frac{2N_\psi}{\sqrt{n!}}\,H_n(\sinh(x))\cosh(x),$$

$n \geq 0$. A straightforward computation shows that these functions are compatible and biorthonormal if $\overline{N_\psi}N_\varphi = \frac{e}{2\sqrt{\pi}}$:

$$\langle\psi_m, \varphi_n\rangle = \delta_{n,m}, \qquad \forall n, m \geq 0.$$

Example 3:

The families $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ are compatible biorthonormal $\mathcal{E}_c$-quasi bases, where

$$\mathcal{E}_c = \left\{ h(s) \in \mathcal{L}^2(\mathbb{R}) : \, h_{[-]}(s) := h(\sinh^{-1}(s))\, e^{s^2/2} \in \mathcal{L}^2(\mathbb{R}) \right\},$$

dense in $\mathcal{L}^2(\mathbb{R})$:

$$\sum_{n=0}^{\infty} \langle f, \varphi_n \rangle \langle \psi_n, g \rangle = \sum_{n=0}^{\infty} \langle f, \psi_n \rangle \langle \varphi_n, g \rangle = \langle f, g \rangle,$$

$\forall f, g \in \mathcal{E}_c$.
Moreover, they are respectively eigenstates of the following operators:

$$H = -\frac{1}{2(\cosh(x))^2} \frac{d^2}{dx^2} + \frac{1}{2} \left((\text{sech}(x))^2 - 2 \right) \tanh(x) \frac{d}{dx} - 1,$$

and

$$H^\dagger = -\frac{1}{2(\cosh(x))^2} \frac{d^2}{dx^2} + \left(\frac{3}{2} (\text{sech}(x))^2 + 1 \right) \tanh(x) \frac{d}{dx} + \\ - \frac{1}{8(\cosh(x))^4} (-9 + 4\cosh(2x) + \cosh(4x)),$$

with eigenvalues $\mathbb{N} \cup \{0\}$.

Bi-coherent states: few general facts

Let us consider two biorthonormal families of vectors, $\mathcal{F}_{\tilde{\varphi}} = \{\tilde{\varphi}_n \in \mathcal{H},\, n \geq 0\}$ and $\mathcal{F}_{\tilde{\Psi}} = \{\tilde{\Psi}_n \in \mathcal{H},\, n \geq 0\}$ which are $\mathcal{G}$ -quasi bases for some dense subset of $\mathcal{H}$, $\mathcal{G}$.

Let $\{\alpha_n\}$ be such that $0 = \alpha_0 < \alpha_1 < \alpha_2 < \ldots$, and $\overline{\alpha} = \sup_n \alpha_n$.

Remark:– For pseudo-bosons $\alpha_n = \sqrt{n}$ and $\overline{\alpha} = \infty$.

We further consider two operators, A and $B^\dagger$, which act as lowering operators respectively on $\mathcal{F}_{\tilde{\varphi}}$ and $\mathcal{F}_{\tilde{\Psi}}$ in the following way:

$$A\,\tilde{\varphi}_n = \alpha_n \tilde{\varphi}_{n-1}, \qquad B^\dagger\,\tilde{\Psi}_n = \alpha_n \tilde{\Psi}_{n-1},$$

for all $n \geq 1$, with $A\,\tilde{\varphi}_0 = B^\dagger\,\tilde{\Psi}_0 = 0$.

Notice that we are not requiring $\tilde{\varphi}_n$ or $\tilde{\Psi}_n$ to have uniformly bounded norms. This is true for o.n. basis, or for Riesz bases, but not in general.

Then we have:

Bi-coherent states: few general facts

Theorem, pt 1:

Assume that four strictly positive constants A_φ, A_Ψ, r_φ and r_Ψ exist, together with two strictly positive sequences $M_n(\varphi)$ and $M_n(\Psi)$, for which

$$\lim_{n\to\infty}\frac{M_n(\varphi)}{M_{n+1}(\varphi)}=M(\varphi),\qquad \lim_{n\to\infty}\frac{M_n(\Psi)}{M_{n+1}(\Psi)}=M(\Psi),$$

where $M(\varphi)$ and $M(\Psi)$ could be infinity, and such that, for all $n\geq 0$,

$$\|\tilde\varphi_n\|\leq A_\varphi\, r_\varphi^n M_n(\varphi),\qquad \|\tilde\Psi_n\|\leq A_\Psi\, r_\Psi^n M_n(\Psi).$$

Then the following series:

$$N(|z|)=\left(\sum_{k=0}^\infty\frac{|z|^{2k}}{(\alpha_k!)^2}\right)^{-1/2},$$

$$\varphi(z)=N(|z|)\sum_{k=0}^\infty\frac{z^k}{\alpha_k!}\tilde\varphi_k,\qquad \Psi(z)=N(|z|)\sum_{k=0}^\infty\frac{z^k}{\alpha_k!}\tilde\Psi_k,$$

are all convergent inside the circle $C_\rho(0)$ in $\mathbb{C}$ centered in the origin of the complex plane and of radius $\rho=\overline{\alpha}\,\min\left(1,\frac{M(\varphi)}{r_\varphi},\frac{M(\Psi)}{r_\Psi}\right)$.

Bi-coherent states: few general facts

Theorem, pt 2:

Moreover, for all $z \in C_\rho(0)$,

$$\langle \varphi(z), \Psi(z) \rangle = 1,$$

$$A\varphi(z) = z\varphi(z), \qquad B^\dagger \Psi(z) = z\Psi(z).$$

Suppose further that a measure $d\lambda(r)$ does exist such that

$$\int_0^\rho d\lambda(r)\, r^{2k} = \frac{(\alpha_k!)^2}{2\pi},$$

for all $k \geq 0$. Then, putting $z = re^{i\theta}$ and calling $d\nu(z, \overline{z}) = N(r)^{-2} d\lambda(r) d\theta$, we have

$$\int_{C_\rho(0)} \langle f, \Psi(z) \rangle \langle \varphi(z), g \rangle \, d\nu(z, \overline{z}) = \int_{C_\rho(0)} \langle f, \varphi(z) \rangle \langle \Psi(z), g \rangle \, d\nu(z, \overline{z}) = \langle f, g \rangle,$$

for all $f, g \in \mathcal{G}$.

Bi-coherent states: few general facts

Some comments:

the norms of the vectors $\tilde{\varphi}_n$ and $\tilde{\Psi}_n$ need not being uniformly bounded, here. On the contrary, they can diverge rather fast with n. This is reflected by the fact that bi-coherent states of this kind only exist inside $C_\rho(0)$.

However, if A and B are pseudo-bosonic, then $\alpha_n = \sqrt{n}$ and, therefore, $\overline{\alpha} = \infty$ and $C_\rho(0)$ coincides with the whole complex plane, at least if $M(\varphi)$ and $M(\Psi)$ are both non zero. Also: the moment problem is easy to solve.

We also observe that no mention is made here to the *displacement-like* operators usually relevant in connection with ordinary coherent states. We are referring here to the unitary operator $U(z) = e^{\overline{z}\,c - zc^\dagger}$, where $[c, c^\dagger] = 1\!\!1$, which should be replaced now, for instance, by $e^{\overline{z}\,A - zB}$. This is, indeed, a non trivial aspect of the theory of bi-coherent states.

Notice that the theorem is given in an Hilbert space. Indeed, $\tilde{\varphi}_n$ and $\tilde{\Psi}_n$ have finite norms, as well as the vectors $\varphi(z)$ and $\Psi(z)$. However, as we have shown before, $\|\tilde{\varphi}_n\| = \|\tilde{\Psi}_n\| = \infty$, for all (or some) n. Hence, working in $\mathcal{H}$ is not the most appropriate choice. This motivates our next discussion...

Weak bi-coherent states

We concentrate here on Example 3 above. We recall that

$$\mathcal{E}_c=\left\{h(s)\in\mathcal{L}^2(\mathbb{R}):\, h_{[-]}(s):=h(\sinh^{-1}(s))\,e^{s^2/2}\in\mathcal{L}^2(\mathbb{R})\right\}.$$

It is useful to notice that, taken $h(s)\in\mathcal{E}_c$, then

$$h_{[+]}(s):=h(\sinh^{-1}(s))\,\frac{e^{-s^2/2}}{\sqrt{1+s^2}}\in\mathcal{L}^2(\mathbb{R}),$$

as well.

A natural topology on $\mathcal{E}_c$ is the following:

Topology $\tau_{\mathcal{E}_c}$ and the dual of $\mathcal{E}_c$

We say that a sequence $\{g_n(x)\}$ in $\mathcal{E}_c$ is $\tau_{\mathcal{E}_c}$-convergent to a certain $g(x)\in\mathcal{L}^2(\mathbb{R})$ if $\{g_n(x)\}$ and $\{(g_n)_{[-]}(x)\}$ converge to $g(x)$ and to $g_{[-]}(x)$ respectively, in the norm $\|.\|$ of $\mathcal{L}^2(\mathbb{R})$. It is clear that, when this is true, $g(x)\in\mathcal{E}_c$. Hence, $\mathcal{E}_c$ is closed in $\tau_{\mathcal{E}_c}$. We call $\mathcal{E}_c'$ the set of all continuous linear functionals on $\mathcal{E}_c$.

Lemma:– if $g_n(x)\in\mathcal{E}_c$ is $\tau_{\mathcal{E}_c}$-convergent to a certain $g(x)$, then $\{(g_n)_{[+]}(x)\}$ converges to $g_{[+]}(x)$ in $\|.\|$.

Weak bi-coherent states

Let us now introduce $\Phi(z)$ and $\Psi(z)$ as follows: $\forall g(x)\in\mathcal{E}_c$ we put

$$\langle\Phi(z),g\rangle=e^{-|z|^2/2}\sum_{n\geq0}\frac{\overline{z}^n}{\sqrt{n!}}\langle\varphi_n,g\rangle,\quad \langle\Psi(z),g\rangle=e^{-|z|^2/2}\sum_{n\geq0}\frac{\overline{z}^n}{\sqrt{n!}}\langle\psi_n,g\rangle.$$

These are well defined, for all $z\in\mathbb{C}$. Indeed we have

$$\langle g,\varphi_n\rangle=\frac{N_\varphi\pi^{1/4}}{\sqrt{2^n}\,e}\langle g_{[+]},e_n\rangle,\qquad \langle\psi_n,g\rangle=2\overline{N}_\psi\sqrt{2^n\sqrt{\pi}}\langle e_n,g_{[-]}\rangle,$$

for all $g(x)\in\mathcal{E}_c$, and therefore, since $\|e_n\|=1$,

$$|\langle g,\varphi_n\rangle|\leq\frac{|N_\varphi|\pi^{1/4}}{\sqrt{2^n}\,e}\|g_{[+]}\|,\qquad |\langle\psi_n,g\rangle|\leq 2|N_\psi|\sqrt{2^n\sqrt{\pi}}\|g_{[-]}\|,$$

for all $n\geq0$. But $\|g_{[+]}\|,\|g_{[-]}\|<\infty$, due to the definition of $\mathcal{E}_c$, and therefore **the series above are always convergent.**

We can now introduce two linear functionals on $\mathcal{E}_c$, $F_\Phi(z)$ and $F_\Psi(z)$, as follows:

$$F_\Phi(z)[g]=\langle\Phi(z),g\rangle,\qquad F_\Psi(z)[g]=\langle\Psi(z),g\rangle,$$

for all $z\in\mathbb{C}$ and $\forall g\in\mathcal{E}_c$. We want to check that they are $\tau_{\mathcal{E}_c}$-continuous and, therefore, define *some sort of distribution*.

Weak bi-coherent states

In fact we can prove the following results:

Proposition:– $F_\Phi(z)$ and $F_\Psi(z)$ belong to $\mathcal{E}_c'$.

We only have to prove that these functionals are $\tau_{\mathcal{E}_c}$-continuous. For that, let us consider a sequence $\{g_n(x) \in \mathcal{E}_c\}$ which is $\tau_{\mathcal{E}_c}$-convergent to $g(x)$. As we have shown this implies that $(g_n)_{[\pm]}(x)$ converges to $g_{[\pm]}(x)$ in $\|.\|$. With easy estimates we conclude that

$$|F_\Phi(z)[g_n - g]| \leq e^{-|z|^2/2}\frac{|N_\varphi|\pi^{1/4}}{e}\left(\sum_{n=0}^{\infty}\frac{|z|^n}{\sqrt{2^n\, n!}}\right)\|(g_k)_{[+]} - g_{[+]}\| \to 0,$$

when $k \to \infty$, for all $z \in \mathbb{C}$. Also,

$$|F_\Psi(z)[g_n - g]| \leq e^{-|z|^2/2}2|N_\psi|\pi^{1/4}\left(\sum_{n=0}^{\infty}\frac{(\sqrt{2}|z|)^n}{\sqrt{n!}}\right)\|(g_k)_{[-]} - g_{[-]}\| \to 0,$$

again for all $z \in \mathbb{C}$ and for $k \to \infty$.

Weak bi-coherent states

We also have

Proposition:– The pair $(\Phi(z),\Psi(z))$ satisfies the following properties:
(i) for all $g(x)\in\mathcal{D}(\mathbb{R})$ we have

$$\langle g,a\Phi(z)\rangle = z\langle g,\Phi(z)\rangle, \qquad \langle g,b^\dagger\Psi(z)\rangle = z\langle g,\Psi(z)\rangle,$$

for all $z\in\mathbb{C}$.

(ii) We have

$$\frac{1}{\pi}\int_{\mathbb{C}}\langle f,\Phi(z)\rangle\,\langle\Psi(z),g\rangle\,dz = \frac{1}{\pi}\int_{\mathbb{C}}\langle f,\Psi(z)\rangle\,\langle\Phi(z),g\rangle\,dz = \langle f,g\rangle\,,$$

for all $f,g\in\mathcal{E}_c$.

This proposition reflects some properties of the (bi-)coherent states, but in a weak sense: they are (weak) eigenstates of the pseudo-bosonic annihilation operators and they produce a resolution of the identity on $\mathcal{E}_c$.

An important comment:– In what we have shown there is no mention to any displacement-like operator, as for ordinary coherent states. Something will be discussed in Part 2.

Displacement-like operators

PART TWO

Prelude: problems with unbounded operators

We want to show what can be wrong with unbounded operators with a very simple example.

Let $\hat{x}$ and $\hat{p}$ be the position and momentum operators, $[\hat{x},\hat{p}]=i\mathbb{1}$. We put

$$c=\frac{1}{\sqrt{2}}(\hat{x}+i\hat{p}) \qquad \Rightarrow \qquad [c,c^\dagger]=\mathbb{1}.$$

The vacuum of c, $c\,e_0(x)=0$, satisfies the differential equation

$$e_0'(x)=-x\,e_0(x) \qquad \Rightarrow \qquad e_0(x)=Ne^{-x^2/2}\in\mathcal{L}^2(\mathbb{R}).$$

Let us consider the operator $T=e^{\frac{7\pi}{8}\left(c^2+(c^\dagger)^2\right)}$. This operator is unbounded, invertible, and (formally) self-adjoint.

Warning:

We want to show that working with T as if it was a bounded operator creates **paradoxes** (similar to others we can find when working formally with the metric operator).

Prelude: problems with unbounded operators

First of all, it is easy to check that

$$\hat{x} = \frac{1}{\sqrt{2}}(c + c^\dagger) = TcT^{-1}.$$

Now, defining $\Phi_0(x) = T\,e_0(x)$, we should have,

$$\hat{x}\,\Phi_0(x) = (TcT^{-1})(T\,e_0(x)) = T\,c\,e_0(x) = 0,$$

which admits as the only (continuous) solution $\Phi_0(x) = 0$. But this is **not compatible** with the existence of T^{-1} and with the fact that $e_0(x) \neq 0$, since

$$0 \neq e_0(x) = T^{-1}\Phi_0(x) = 0.$$

Incidentally we observe that a non trivial solution of $x\,\Phi_0(x) = 0$ does exist, but only in a distributional sense:

$$\Phi_0(x) = N'\delta(x).$$

Prelude: problems with unbounded operators

The same conclusion can be deduced by noticing that, with a little algebra,

$$T\,e_0 = e^{-\frac{1}{2}c^{\dagger\,2}}\, e^{\frac{1}{4}\log 2(cc^\dagger + c^\dagger c)}\, e^{-\frac{1}{2}c^2}\, e_0 =$$

$$= 2^{1/4} e^{-\frac{1}{2}c^{\dagger\,2}}\, e_0 = 2^{1/4}\sum_{k=0}^{\infty}\frac{\sqrt{2k!}}{k!}\left(-\frac{1}{2}\right)^k e_{2k}.$$

Here $\{e_k\}_{k\geq 0}$ is the basis of $\mathcal{L}^2(\mathbb{R})$ made by the eigenstates of the quantum harmonic oscillator. It is possible to check that the series for $\|T\,e_0\|^2$ diverges. Hence $T\,e_0$ is not a vector in $\mathcal{L}^2(\mathbb{R})$, in agreement with what we have explicitly deduced above: $e_0 \notin D(T)$!

What we learn?

We learn that, in presence of unbounded operators, what is *reasonable* is not necessarily *true* !!

The displacement operator and the BCH formula

If A and B are two *operators* (bounded or not, for the moment) such that $[A, B]$ commutes with both A and B, then the BCH formula states that

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} = e^B e^A e^{\frac{1}{2}[A,B]}.$$

The various parts of this equations are surely well defined if A and B are both bounded. In this case, in fact, e^A and e^B are bounded, too, since they can be represented by norm-convergent series. Then $e^B e^A$ and $e^A e^B$ are also both well defined. But, if A and B are not bounded, then there is no reason a priori to be sure that:

(1) e^A and e^B exist (except if $A = A^\dagger$ and $B = B^\dagger$: spectral theorem!);

(2) the domain of e^A, $D(e^A)$, is not empty and, even if this is true and taken $0 \neq f \in D(e^A)$, then $e^A f \in D(e^B)$ (i.e., $e^B e^A f$ might not exist);

(3) $D(e^B)$ is not empty and, taken $f \in D(e^B)$, $f \neq 0$, then $e^B f \in D(e^A)$ (i.e., $e^A e^B f$ might not exist);

(4) e^{A+B} exists.

(5) Last but not least, also if all these aspects can be solved positively, still there is no reason to be sure that the equalities above are satisfied, and in which sense.

The displacement operator and the BCH formula

This is exactly the case for the displacement operator

$$D(z) = e^{zc^\dagger - \overline{z}\, c},$$

$z \in \mathbb{C}$ and $[c, c^\dagger] = 1$. There is no doubt that

$$D(z) = e^{zc^\dagger - \overline{z}\, c} = e^{zc^\dagger} e^{-\overline{z}\, c} e^{-\frac{1}{2}|z|^2} = e^{-\overline{z}\, c} e^{zc^\dagger} e^{\frac{1}{2}|z|^2}$$

works! It has been used thousands of time, with success. But...why? And in which sense the above equality should be understood?

It is clear that $D(z)$ is well defined in all of $\mathcal{H}$, being unitary. But what about $e^{zc^\dagger}$ and $e^{-\overline{z}\, c}$. And what about their products?

Obviously this factorization problem does not exist if the displacement operator is rewritten in terms of the position and the momentum operators $\hat{x}$ and $\hat{p}$ since, in this case, the formula above is replaced by

$$e^{i(\alpha\hat{x}+\beta\hat{p})} = e^{i\,\frac{\alpha\beta}{2}} e^{i\alpha\hat{x}} e^{i\beta\hat{p}},$$

with $\alpha, \beta \in \mathbb{R}$, which only involves unitary operators. Still, this formula needs some deep mathematics to be proved.

The displacement operator and the BCH formula

Remarks:– (1) For bi-coherent states rather than $D(z)$ one is interested in, e.g.,

$$D(z,w) = e^{zc^\dagger - \overline{w}\,c}, \qquad \text{or} \qquad V(z) = e^{zb - \overline{z}a},$$

$[a,b] = \mathbb{1}$, which are no longer unitary. More problems!

(2) Only few books (in my knowledge) deal with these mathematical aspects of $D(z)$:
M. Combescure and R. Didier, *Coherent states and applications in mathematical physics*, Springer (2012), and
B. Hall, *Quantum theory for mathematicians*, Springer (2013)

Mine is a brute force approach. I use estimates (MANY ESTIMATES) to check the existence of all the operators I need, which I can check are well defined on the dense domain of all the finite linear combinations of the e_n's, $\mathcal{L}_e$, (the usual eigenvectors of the bosonic number operator).

I prefer to avoid using, e.g., the spectral theorem since this holds for self-adjoint operators, while the operators I am interested in are often **not** self-adjoint!

The displacement operator and the BCH formula

The main steps:

1. we define $e^{\alpha c}$ and $e^{\beta c^\dagger}$, $\alpha, \beta \in \mathbb{C}$, on $\mathcal{L}_e$;
2. we prove that $e^{\alpha c}$ and $e^{\beta c^\dagger}$ can be multiplied, in any order: $e^{\alpha c} e^{\beta c^\dagger} f$ and $e^{\beta c^\dagger} e^{\alpha c} f$ both exist for all $\alpha, \beta \in \mathbb{C}$ and $\forall f \in \mathcal{L}_e$;
3. we prove that the order can be exchanged, adding an extra factor, as in $e^{\beta c^\dagger} e^{\alpha c} f = e^{-\alpha\beta} e^{\alpha c} e^{\beta c^\dagger} f$, $f \in \mathcal{L}_e$;
4. we prove that $e^{\alpha c + \beta c^\dagger} f$ is well defined, $f \in \mathcal{L}_e$;
5. we prove the BCH formula in this form: $e^{\alpha c} e^{\beta c^\dagger} f = e^{\frac{1}{2}\alpha\beta} e^{\alpha c + \beta c^\dagger} f$, $f \in \mathcal{L}_e$.

Along the way several other technical useful results are proved. Most of these results are well known but, to my knowledge, only few are proved rigorously.

For instance: how to define $e^{\alpha c + \beta c^\dagger}$?

The displacement operator and the BCH formula

The starting point if the following estimates:

$$\left\|(\alpha c+\beta c^\dagger)^l\, e_k\right\| \leq \sqrt{\frac{(k+l)!}{k!}}\,(|\alpha|+|\beta|)^l\,,$$

$\forall\alpha,\beta\in\mathbb{C}$, and $\forall k\geq 0$ fixed and for $l=0,1,2,3,\ldots$. Its proof goes by induction on l.

Then we have

$$\left\|\sum_{l=0}^{\infty}\frac{1}{l!}(\alpha c+\beta c^\dagger)^l e_k\right\| \leq \sum_{l=0}^{\infty}\frac{1}{l!}\sqrt{\frac{(k+l)!}{k!}}\,(|\alpha|+|\beta|)^l\,,$$

for all $k\geq 0$, and the RHS is a power series in $|\alpha|+|\beta|$. The radius of convergence is easily computed and in $\rho=\infty$, for all k. This implies that the series in the LHS converges and defines $e^{\alpha c+\beta c^\dagger}e_k$. We can extend the definition to $\mathcal{L}_e$ by linearity. Since $\mathcal{L}_e$ is dense in $\mathcal{H}$, $e^{\alpha c+\beta c^\dagger}$ is densely defined.

More results

The existence of the displacement-like operators for PBs is proved also, using a similar strategy.
In other words, we can use similar estimates also when $(c, c^\dagger)$ are replaced by the pseudo-bosonic operators (a, b).
This allows us to check equalities like

$$e^{zb-\overline{z}\,a}\varphi_0 = e^{-\frac{1}{2}|z|^2}e^{z\,b}e^{-\overline{z}\,a}\varphi_0 = e^{-\frac{1}{2}\,|z|^2}\sum_{k=0}^{\infty}\frac{z^k}{\sqrt{k!}}\varphi_k,$$

and

$$e^{za^\dagger-\overline{z}\,b^\dagger}\psi_0 = e^{-\frac{1}{2}|z|^2}e^{z\,a^\dagger}e^{-\overline{z}\,b^\dagger}\psi_0 = e^{-\frac{1}{2}\,|z|^2}\sum_{k=0}^{\infty}\frac{z^k}{\sqrt{k!}}\psi_k.$$

These equalities extend those for ordinary coherent states to PBs.

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