A real expectation value of the time-dependent non-Hermitian Hamiltonians

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- $\mathcal{PT}$**-Symmetry (Carl Bender et al, 1998)**

A non-hermitian Hamiltonian H $(H^{\dagger} \neq H)$ is $\mathcal{PT}$-symmetric if it is invariant under the transformation $\mathcal{PT}$

$$H = (\mathcal{PT})\, H\, (\mathcal{PT}) \quad \Leftrightarrow \quad [H, \mathcal{PT}] = 0.$$

where $\mathcal{P}$ is the parity operator and $\mathcal{T}$ is the reversal time operator.
As a consequence the eingenvalues of a PT-Symmetric Hamiltonian are **REALS.**

- $\mathcal{PT}$**-inner Product**

The $\mathcal{PT}$ inner-product associated to $\mathcal{PT}$-symmetric Hamiltonians is defined as

$$\langle f, g\rangle_{\mathcal{PT}} = \int_C dx \left[\mathcal{PT} f(x)\right] g(x), \tag{1}$$

where $\mathcal{PT} f(x) = f^*(-x)$. The application of this definition to the eigenfunctions of H and $\mathcal{PT}$ implies

$$\langle \psi_m, \psi_n\rangle_{\mathcal{PT}} = (-1)^n \delta_{mn}, \tag{2}$$

where the norm is not always definite positive.

- $\mathcal{CPT}$**-inner Product**

To solve the problem of negative norm, Bender et al (2002) introduced a new operator $\mathcal{C}$ with eigenvalues ± 1 such that $\mathcal{C}$

$$\mathcal{C}^2 = 1, \quad [\mathcal{C}, \mathcal{PT}] = 0, \quad [\mathcal{C}, \mathcal{P}] \neq 0, \quad [\mathcal{C}, \mathcal{T}] \neq 0,$$

Then, a new $\mathcal{CPT}$-inner-product has been defined

$$\langle f, g \rangle_{\mathcal{CPT}} = \int_C dx \, [\mathcal{CPT} f(x)] \, g(x), \tag{3}$$

such the norm becomes positive

$$\langle \psi_m, \psi_n \rangle_{\mathcal{CPT}} = \delta_{mn} \ . \tag{4}$$

In order to obtain the solution of the Schrödinger equation with a non-hermitian time-dependent (TD) Hamiltonian $H(t)$

$$i\frac{\partial}{\partial t}\left|\psi(t)\right\rangle = H(t)\left|\psi(t)\right\rangle, \tag{5}$$

we seek that $H(t)$ can be converted into a time-independent Hamiltonian via TD-transformations. To this end, we perform a unitary transformation $F(t)$ on $\left|\psi(t)\right\rangle$

$$\left|\chi(t)\right\rangle = F(t)\left|\psi(t)\right\rangle, \tag{6}$$

by inserting (6) in Eq. (5), the Schrödinger equation for the state $\left|\chi(t)\right\rangle$ is

$$i\frac{\partial}{\partial t}\left|\chi(t)\right\rangle = \mathcal{H}\left|\chi(t)\right\rangle, \tag{7}$$

such that the new Hamiltonian $\mathcal{H}$

$$\mathcal{H} = F(t)H(t)F^{+}(t) - iF(t)\frac{\partial F^{+}(t)}{\partial t}, \tag{8}$$

is time-independent and $\mathcal{PT}$-symmetric

$$\mathcal{H} \equiv \mathcal{H}_0^{\mathcal{PT}}. \tag{9}$$

Its eigenstates $|\chi(t)\rangle$ preserve the $\mathcal{CPT}$-inner product

$$\langle\chi(t)|\,\chi(t)\rangle_{\mathcal{CPT}} = \langle\chi(t)|\,\mathcal{CP}\,|\chi(t)\rangle\,, \tag{10}$$

and in this case the solution of the Schrödinger equation (7) can be written as

$$|\chi(t)\rangle = \exp(-iEt)\,|\chi\rangle\,. \tag{11}$$

where $|\chi\rangle$ is an eigenstate of $\mathcal{H}_0^{\mathcal{PT}}$.

Expectation value of non-Hermitian time-dependent Hamiltonian

We calculate firstly the expectation value of the Hamiltonian $\mathcal{H}_0^{\mathcal{PT}}$

$$\left\langle \mathcal{H}_0^{\mathcal{PT}} \right\rangle_{\mathcal{CPT}} = \langle \chi(t) | \mathcal{CP} \mathcal{H}_0^{\mathcal{PT}} | \chi(t) \rangle \quad (12)$$

$$= \langle \chi(t) | \mathcal{CP} \left[F H(t) F^{+} - iF \frac{\partial F^{+}}{\partial t} \right] | \chi(t) \rangle , \quad (13)$$

From which we deduce that

$$\langle \chi(t) | \mathcal{CP} \left[F H(t) F^{+} \right] | \chi(t) \rangle = \left\langle \mathcal{H}_0^{\mathcal{PT}} \right\rangle_{\mathcal{CPT}} + \langle \chi(t) | \mathcal{CP} \left[iF \frac{\partial F^{+}}{\partial t} \right] | \chi(t) \rangle . \quad (14)$$

The first term is the expectation value of $H(t)$ with a new $\mathcal{C}(t)\mathcal{PT}$-inner product

$$\langle \chi(t) | \mathcal{CP} \left[F H(t) F^{+} \right] | \chi(t) \rangle = \langle \chi(t) | (FF^{+}) \mathcal{C} (FF^{+}) \mathcal{P} \left[F H(t) F^{+} \right] | \chi(t) \rangle$$

$$= \langle \psi(t) | \mathcal{C}(t) \mathcal{P} H(t) | \psi(t) \rangle = \langle H(t) \rangle_{\mathcal{C}(t)\mathcal{PT}} , \quad (15)$$

Expectation value of non-Hermitian time-dependent Hamiltonian

where $[\mathcal{P}, F] = 0$ and the new operator $\mathcal{C}(t)$ is defined as

$$\mathcal{C}(t) = F^{+}(t)\mathcal{C}F(t)$$

which is similar to the $\mathcal{C}$ operator in the sense that verifies the properties

$$\mathcal{C}^2(t) = 1 \;\text{ since } \mathcal{C}^2 = 1, \;\text{ and }\; [\mathcal{C}(t), \mathcal{PT}] \neq 0,$$

Finally

$$\langle H(t)\rangle_{\mathcal{C}(t)\mathcal{PT}} = \left\langle \mathcal{H}_0^{\mathcal{PT}} \right\rangle_{\mathcal{CPT}} + \langle\chi(t)|\,\mathcal{CP}\left[iF\frac{\partial F^{+}}{\partial t}\right]|\chi(t)\rangle\,. \qquad (16)$$

Indeed, since $\mathcal{H}_0^{\mathcal{PT}}$ is $\mathcal{PT}$ symmetric and F is unitary, the expectation value $\langle H(t)\rangle_{\mathcal{C}(t)\mathcal{PT}}$ **is guaranteed to be REAL**.

Let us consider the non-Hermitian Hamiltonian of a class of 1D time-dependent harmonic oscillators with variable mass $m(t) = m_0\alpha(t)$ subjected to a driving linear complex time-dependent potential in the form $i\lambda(t)x$

$$H(t) = \frac{p^2}{2m_0\alpha(t)} + \alpha(t)\frac{m_0\omega^2}{2}x^2 + i\lambda(t)x, \tag{17}$$

where $\alpha(t)$ is a positive real time-dependent function, x and p are the canonical conjugates $[x, p] = i$. The mass m_0 and the frequency ω are the characteristic parameters of the quantum system.

Application

In order to solve the TD Schrödinger equation with the above Hamiltonian can be obtained by introducing two consecutive unitary transformations F_1 and F_2, where $[\mathcal{P}, F_1] = [\mathcal{P}, F_2] = 0$.The first is

$$F_1(t) = \exp\left[-\frac{i}{2}\{x, p\} \ln\left(\sqrt{\alpha(t)}\right)\right] \tag{18}$$

where the function $\lambda(t)$ in the complex potential should be $\lambda(t) = \sqrt{\alpha(t)}$ in order to obtain in Eq. (8) a time-independent $\mathcal{PT}$-symmetric Hamiltonian $\mathcal{H}_0^{\mathcal{PT}}$.

$$H(t) = \frac{p^2}{2m_0\alpha(t)} + \alpha(t)\frac{m_0\omega^2}{2}x^2 + ix\sqrt{\alpha(t)}, \tag{19}$$

Application

The unitary operator $F_1(t)$ has the properties

$$F_1 x F_1^{+} = \frac{x}{\sqrt{\alpha(t)}}, \qquad F_1 p F_1^{+} = p\sqrt{\alpha(t)} \ ,$$

In the x-representation, the wave function is given by

$$\langle x| F_1 |\phi\rangle = \alpha^{-\frac{1}{2}} \phi \left(x \alpha^{-\frac{1}{2}} \right)$$

Suppose that

$$|\phi(t)\rangle = F_1(t) |\psi(t)\rangle \ , \tag{20}$$

Substituting (20) into (5) ruled by the Hamiltonian (19), we find the equation of motion for $|\phi(t)\rangle$

$$i \frac{\partial}{\partial t} |\phi(t)\rangle = H_1(t) |\phi(t)\rangle \ , \tag{21}$$

Application

where the Hamiltonian

$$H_1(t) = F_1(t)H(t)F_1^+(t) - iF_1(t)\frac{\partial F_1^+(t)}{\partial t}$$
$$= \frac{p^2}{2m_0} + \frac{m_0\omega^2}{2}x^2 + ix + \frac{1}{4}\frac{\dot{\alpha}(t)}{\alpha(t)}(xp + px) \qquad (22)$$

which look like the time-independent harmonic oscillators with constant mass m_0 subjected to a driving linear complex time-independent potential plus a time dependent $(xp + px)$ terms. In order to obtain the usual time-dependent harmonic oscillator with a perturbative linear potential, we remove the cross term in (22) via the second transformation

$$F_2(t) = \exp\left[i\frac{m_0\dot{\alpha}(t)}{4\alpha(t)}x^2\right], \qquad (23)$$

Application

where its properties are

$$F_2 x F_2^+ = x, \qquad F_2 p F_2^+ = -\frac{m_0 \dot{\alpha}(t)}{2\alpha(t)} x \ ,$$

Thus, the following unitary transformation $F(t) = F_2(t) F_1(t)$

$$F(t) = \exp\left[i \frac{m_0 \dot{\alpha}(t)}{4\alpha(t)} x^2\right] \exp\left[-\frac{i}{2}\{x, p\} \ln\left(\sqrt{\alpha(t)}\right)\right], \tag{24}$$

transforms the canonical operators x and p and their squares x^2 and p^2 as follows

$$F x F^+ = \frac{x}{\sqrt{\alpha(t)}}, \qquad F p F^+ = p\sqrt{\alpha(t)} - \frac{m_0 \dot{\alpha}(t)}{2\sqrt{\alpha(t)}} x \ ,$$

$$F p^2 F^+ = \alpha(t) x^2 - \frac{1}{2} m_0 \dot{\alpha}(t) \{x, p\} + \frac{m_0^2 \dot{\alpha}^2(t)}{4\alpha(t)} x^2, \quad F x^2 F^+ = \frac{x^2}{\alpha(t)}, \tag{25}$$

Application

Thus, the transformed Hamiltonian (8) reads

$$\mathcal{H} = \frac{p^2}{2m_0} + \frac{1}{2} m_0 \left(\omega^2 + \frac{1}{4} \frac{\dot{\alpha}^2(t)}{\alpha^2(t)} - \frac{\ddot{\alpha}(t)}{2\alpha(t)} \right) x^2 + ix. \qquad (26)$$

Now we require that the Hamiltonian (26) governing the evolution of $|\chi(t)\rangle$ is a simple time-independent harmonic oscillator with constant mass m_0 subjected to a driving linear complex time-independent potential. Then, let us set the time-dependent frequency appearing in Eq. (26) equal to a real constant denoted Ω_0^2 so that the resulting non-Hermitian Hamiltonian is $\mathcal{PT}$-symmetric :

$$\omega^2 + \frac{1}{4} \frac{\dot{\alpha}^2(t)}{\alpha^2(t)} - \frac{\ddot{\alpha}(t)}{2\alpha(t)} = \Omega_0^2 \ = \text{Constant}, \qquad (27)$$

which is an auxiliary equation

$$\ddot{\alpha} - \frac{\dot{\alpha}^2}{2\alpha} + 2\alpha\left(\Omega_0^2 - \omega^2\right) = 0, \tag{28}$$

therefore, the expression of the $\mathcal{PT}$-symmetric time-independent Hamiltonian $\mathcal{H}_0^{\mathcal{PT}}$ is

$$\mathcal{H}_0^{\mathcal{PT}} = \frac{p^2}{2m_0} + \frac{1}{2}m_0\Omega_0^2 x^2 + ix, \tag{29}$$

where Ω_0 can be determined from the solution $\alpha(t)$ of Eq. (28). By making the change $\alpha(t) = \frac{1}{\rho^2(t)}$, the auxiliary equation (28) is transformed to the new form

$$\ddot{\rho} + \left(\Omega_0^2 - \omega^2\right)\rho = 0, \tag{30}$$

Application

which admits the two solutions :

• For $\Omega_0^2 > \omega^2$: $\rho(t) = A\exp\left(it\sqrt{\Omega_0^2 - \omega^2}\right) + B\exp\left(-it\sqrt{\Omega_0^2 - \omega^2}\right)$.
For an appropriate choice of the constants : $A = B$, we obtain the expression of $\alpha(t)$ as $\alpha(t) = \frac{1}{A^2\cos^2\left(t\sqrt{\Omega_0^2 - \omega^2}\right)}$.

• For $\Omega_0^2 < \omega^2$: $\rho(t) = A\exp\left(t\sqrt{\omega^2 - \Omega_0^2}\right) + B\exp\left(-t\sqrt{\omega^2 - \Omega_0^2}\right)$.
For an appropriate choice of the constants : $A = B$, we obtain the expression of $\alpha(t)$ as $\alpha(t) = \frac{1}{A^2\cosh^2\left(t\sqrt{\omega^2 - \Omega_0^2}\right)}$, and when $B = 0$, $A \neq 0$, the expression of $\alpha(t)$ is $\alpha(t) = \frac{1}{A^2}\exp\left(-2t\sqrt{\omega^2 - \Omega_0^2}\right)$ and the Hamiltonian $H(t)$ corresponds to the Caldirola-Kanai oscillator.

The eigenvalue equation of the $\mathcal{PT}$ -symmetric Hamiltonian $\mathcal{H}_0^{\mathcal{PT}}$ has the form

$$\mathcal{H}_0^{\mathcal{PT}} |\chi_n(x)\rangle = E_n |\chi_n(x)\rangle , \tag{31}$$

and the corresponding solution of the Schrödinger equation (7) can be written as

$$|\chi_n(x,t)\rangle = \exp(-iE_n t) |\chi_n(x)\rangle . \tag{32}$$

Let us introduce a non unitary transformation of the form

$$U = \exp\left[-\frac{p}{m_0\Omega_0^2}\right], \text{ such that } \quad |\chi_n(x)\rangle = U |\varphi_n(x)\rangle . \tag{33}$$

The action of U maps the $\mathcal{PT}$-symmetric Hamiltonian $H_0^{\mathcal{PT}}$ to a Hermitian one as

$$h = U^{-1}\mathcal{H}_0^{\mathcal{PT}}U = \frac{p^2}{2m_0} + \frac{m_0\Omega_0^2}{2}x^2 - \frac{1}{2m_0\Omega_0^2}, \tag{34}$$

where the eigenfunctions $|\varphi_n(x)\rangle$ of the Hermitian Hamiltonian h are

$$|\varphi_n(x)\rangle = \left[\frac{\sqrt{m_0\Omega_0}}{n!2^n\sqrt{\pi\hbar}}\right]^{1/2} \exp\left(-\frac{m_0\Omega_0}{2\hbar}x^2\right) H_n\left[x\left(\frac{m_0\Omega_0}{\hbar}\right)^{1/2}\right]. \tag{35}$$

Then, the solutions $|\chi_n(x,t)\rangle$ are

$$|\chi_n(x,t)\rangle = \exp(-iE_n t)U|\varphi_n(x)\rangle,$$

$$|\chi_n(x,t)\rangle = \left[\frac{\sqrt{m_0\Omega_0}}{n!2^n\sqrt{\pi\hbar}}\right]^{1/2} \exp(-iE_n t)\exp\left[-\frac{p}{m_0\Omega_0^2}\right] \exp\left(-\frac{m_0\Omega_0}{2\hbar}x^2\right) H_n\left[\left(\frac{m_0\Omega_0}{\hbar}\right)^{1/2} x\right], \tag{36}$$

where H_n is the Hermite polynomial and the real eigenvalues of the $\mathcal{PT}$ -symmetric Hamiltonian $\mathcal{H}_0^{\mathcal{PT}}$ are

$$E_n = \hbar\Omega_0\left(n+\frac{1}{2}\right) - \frac{1}{2m_0\Omega_0^2}. \tag{37}$$

If we choose the charge conjugation operator $\mathcal{C}$ in the form

$$\mathcal{C} = \exp\left[\frac{2p}{m_0\Omega_0^2}\right]\mathcal{P}, \tag{38}$$

the $\mathcal{CPT}$-inner product is conserved

$$\begin{aligned}\left\langle\chi_n(x,t)\,|\chi_n(x,t)\right\rangle_{\mathcal{CPT}} &= \left\langle\chi_n(x)\right|\mathcal{CP}\left|\chi_n(x)\right\rangle = \left\langle\varphi_n\right|U\mathcal{CP}U\left|\varphi_n\right\rangle \\ &= \left\langle\varphi_n(x)\,|\varphi_n(x)\right\rangle = 1.\end{aligned} \tag{39}$$

Now it is not difficult to calculate the expectation value $\langle H(t)\rangle_{\mathcal{C}(t)\mathcal{PT}}$ of the Hamiltonian $H(t)$ defined as

$$\langle H(t)\rangle_{\mathcal{C}(t)\mathcal{PT}} = E_n - \frac{\dot{\alpha}(t)}{4\alpha(t)}\left\langle \varphi_n(x)\right| U^{-1}\{x,p\}\,U\left|\varphi_n(x)\right\rangle \tag{40}$$

$$+\left(\frac{m_0\ddot{\alpha}(t)}{4\alpha(t)}\right)\left\langle \chi_n(x)\right|\mathcal{CP}x^2\left|\chi_n(x)\right\rangle, \tag{41}$$

$$\langle H(t)\rangle_{\mathcal{C}(t)\mathcal{PT}} = E_n - \frac{\dot{\alpha}(t)}{4\alpha(t)}\left\langle \varphi_n(x)\right|\{x,p\}\left|\varphi_n(x)\right\rangle \tag{42}$$

$$+\frac{\dot{\alpha}(t)}{2\alpha(t)}\frac{i}{m_0\Omega_0^2}\left\langle \varphi_n(x)\right| p\left|\varphi_n(x)\right\rangle + \left(\frac{m_0\ddot{\alpha}(t)}{4}\right)\left\langle x^2\right\rangle_{\mathcal{CPT}}, \tag{43}$$

where $\left\langle x^2 \right\rangle_{\mathcal{CPT}} = \left\langle \chi_n(x) \right| \mathcal{CP} x^2 \left| \chi_n(x) \right\rangle$. By using the following relations

$$\left\langle \varphi_n(x) \right| x \left| \varphi_n(x) \right\rangle = \left\langle \varphi_n(x) \right| p \left| \varphi_n(x) \right\rangle = 0, \tag{44}$$

$$\left\langle \varphi_n(x) \right| x^2 \left| \varphi_n(x) \right\rangle = \frac{\hbar}{m_0 \Omega_0} \left(n + \frac{1}{2} \right), \tag{45}$$

$$\left\langle \varphi_n(x) \right| p^2 \left| \varphi_n(x) \right\rangle = m_0 \Omega_0 \hbar \left(n + \frac{1}{2} \right), \tag{46}$$

$$\left\langle \varphi_n(x) \right| \{x, p\} \left| \varphi_n(x) \right\rangle = 0, \tag{47}$$

$$\left\langle x^2 \right\rangle_{\mathcal{CPT}} = \frac{\hbar}{m_0 \Omega} \left(n + \frac{1}{2} \right) - \frac{1}{\left(m_0 \Omega_0^2 \right)^2}, \tag{48}$$

we get the expectation value of $H(t)$ as

$$\langle H(t)\rangle_{\mathcal{C}(t)\mathcal{PT}} = E_n + \left(\frac{m_0\ddot{\alpha}(t)}{4\alpha(t)}\right) < x^2 >_{\mathcal{CPT}} \tag{49}$$

$$= E_n + \frac{\ddot{\alpha}(t)}{4\alpha(t)}\left[\frac{\hbar}{\Omega_0}\left(n+\frac{1}{2}\right) - \frac{1}{m_0\Omega_0^4}\right], \tag{50}$$

which is **REAL** for any positive real time-dependent function $\alpha(t)$ and more simple than the result given in Eq. (28) of the article by A. de Sousa Dutra et al (EPL, 2005) with less constraints on the parameters of the problem.

Now, we calculate the expectation values $\langle x\rangle_{\mathcal{C}(t)\mathcal{PT}}$, $\langle x^2\rangle_{\mathcal{C}(t)\mathcal{PT}}$, $\langle p\rangle_{\mathcal{C}(t)\mathcal{PT}}$ and $\langle p^2\rangle_{\mathcal{C}(t)\mathcal{PT}}$ in the states $\psi_n(x,t)$ of $H(t)$ defined in Eq.(19). In the same way, using the $\mathcal{CPT}$-inner product (39) and after straightforward calculation we obtain that

$$\langle x\rangle_{\mathcal{C}(t)\mathcal{PT}} = \langle\psi_n(x,t)|\, F^{+}\mathcal{CP}Fx\,|\psi_n(x,t)\rangle = -\frac{i}{m_0\Omega^2}\frac{1}{\sqrt{\alpha(t)}}, \tag{51}$$

$$\langle x^2\rangle_{\mathcal{C}(t)\mathcal{PT}} = \langle\psi_n(x,t)|\, F^{+}\mathcal{CP}Fx^2\,|\psi_n(x,t)\rangle \tag{52}$$

$$= \left(n+\frac{1}{2}\right)\frac{\hbar}{m_0\Omega\alpha(t)} - \frac{1}{\alpha(t)\,(m_0\Omega^2)^2}, \tag{53}$$

$$\langle p\rangle_{\mathcal{C}(t)\mathcal{PT}} = \langle\psi_n(x,t)|\, F^{+}\mathcal{CP}Fp\,|\psi_n(x,t)\rangle = \frac{i}{2\Omega^2}\frac{\dot{\alpha}(t)}{\sqrt{\alpha(t)}}, \tag{54}$$

$$\left\langle p^2 \right\rangle_{\mathcal{C}(t)\mathcal{PT}} = \left\langle \psi_n(x,t) \right| F^{+}\mathcal{CP}Fp^2 \left| \psi_n(x,t) \right\rangle ,$$

$$\left\langle p^2 \right\rangle_{\mathcal{C}(t)\mathcal{PT}} = \hbar\Omega \left(n + \frac{1}{2}\right) m_0 \alpha(t) \tag{55}$$

$$+ \left(\frac{m_0 \dot{\alpha}(t)}{2}\right)^2 \left[\left(n + \frac{1}{2}\right) \frac{\hbar}{m_0 \Omega \alpha(t)} - \frac{1}{\alpha(t)\left(m_0 \Omega^2\right)^2}\right] . \tag{56}$$

We calculate also the position and momentum uncertainties

$$\Delta x = \sqrt{\langle x^2 \rangle_{\mathcal{C}(t)\mathcal{PT}} - \langle x \rangle^2_{\mathcal{C}(t)\mathcal{PT}}} = \left[\frac{\hbar}{m_0 \Omega_0 \alpha(t)} \left(n + \frac{1}{2}\right)\right]^{1/2} , \tag{57}$$

$$\Delta p = \sqrt{\langle p^2 \rangle_{\mathcal{C}(t)\mathcal{PT}} - \langle p \rangle^2_{\mathcal{C}(t)\mathcal{PT}}} = \frac{1}{\Delta x}\left[\left(n+\frac{1}{2}\right)^2 + \left(\frac{m_0\dot{\alpha}(t)}{2}\right)^2 \Delta x^4\right]^{1/2} . \tag{58}$$

Thus, the uncertainty product is given by

$$\Delta x \Delta p = \left(n+\frac{1}{2}\right)\sqrt{1+\left(\frac{\hbar\dot{\alpha}(t)}{2\Omega_0\alpha(t)}\right)^2}, \tag{59}$$

it is easy to check that the uncertainty product is always real and greater than or equal to $\frac{1}{2}$ and, consequently, it is physically acceptable for any value of n. The uncertainty product takes the minimal value $\Delta x \Delta p = \frac{1}{2}$ only for $n = 0$ and $\alpha(t)$ =constant (for time independent mass oscillators).

Finally, the probability density is

$$\left|U^{-1}F\psi_n(x,t)\right|^2 = \left|U^{-1}\chi_n(x,t)\right|^2 = |\varphi(x)|^2 = \varphi_n^*(x)\varphi_n(x), \tag{60}$$

$$\left|U^{-1}F\psi_n(x,t)\right|^2 = \left[\frac{\sqrt{m_0\Omega}}{n!2^n\sqrt{\pi\hbar}}\right]\exp\left(-\frac{m_0\Omega}{\hbar}x^2\right)\left(H_n\left[\left(\frac{m_0\Omega}{\hbar}\right)^{1/2}x\right]\right)^2 \tag{61}$$

and therefore

$$\int_{-\infty}^{+\infty}\left|U^{-1}F\psi_n(x,t)\right|^2 dx = 1. \tag{62}$$

Then, for this exemple, the probability density of the wavefunction $\psi_n(x,t)$ of $H(t)$ is the same as the probability density of the eigenstate $\chi_n(x,t)$ of $\mathcal{H}_0^{\mathcal{PT}}$ which is also equal to the probability density of $\varphi_n(x)$ of the standard harmonic oscillator (34).

Conclusion

- In order to obtain the analytical solution of the Schrödinger equation of a time-dependent non-Hermitian Hamiltonian $H(t)$, we search a unitary transformation $F(t)$ that reduces the Hamiltonian $H(t)$ to a time-independent $\mathcal{PT}$-symmetric Hamiltonian $\mathcal{H}_0^{\mathcal{PT}}$.

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- The most important step to have a positive-definite inner product is to find a new operator, which we call $\mathcal{C}(t) = F^{+}(t)\mathcal{C}F(t)$, such that the norm is conserved for the initial system described by the solution $|\psi(t)\rangle$.

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- The main result is that the expectation value of the time-dependent non-Hermitian Hamiltonian $H(t)$ **is REAL** in the new $\mathcal{C}(t)\mathcal{PT}$-inner product since the transformation $F(t)$ is unitary and $[\mathcal{P}, F] = 0$.

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- The main result is that the expectation value of the time-dependent non-Hermitian Hamiltonian $H(t)$ **is REAL** in the new $\mathcal{C}(t)\mathcal{PT}$-inner product since the transformation $F(t)$ is unitary and $[\mathcal{P}, F] = 0$.
- As an illustration, we have investigated a class of quantum time-dependent mass oscillators with a complex linear driving force. The expectation value of the Hamiltonian, the uncertainty relation and probability density have been calculated.

Thank you for your attention