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NEW CLASSES OF DYNAMIC SYSTEMS WITH BENIGN GHOSTS

PHHQP seminar

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MOTIVATION:

... To myself, I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary whilst the great ocean of truth lay all undiscovered before me.

Isaac Newton

• The ocean is now charted up to $E \lesssim 10^3$ GeV, $l \gtrsim 10^{-17}$ cm.
• But it extends up to $M_P \approx 10^{19}$ GeV. We have now explored its 10^{-16}-th part.

PROBLEMS IN QUANTUM (AND CLASSICAL) GRAVITY:

- Nonrenormalizability
- Non-causality. Closed time loops. Paradoxes.

TOE = strings?

- No fundamental quantum string theory
- No phenomenological successes.

An alternative (dream) solution: [A.S., 2005]

Our Universe as a soap film in a flat higher dimensional bulk. The TOE is a field theory in this bulk. Gravity etc is an effective theory living on the film, like

$$H_{\mathrm{soap}} = \sigma \mathcal{A} = \sigma \int d^2x \sqrt{g}$$

TRY

$$S \; = \; -\frac{1}{2h^2}\int \mathrm{Tr}\{F_{MN}F_{MN}\}\, d^6x\,,$$

in $D = 6,\ M, N = 0, 1, 2, 3, 4, 5.$

- Dimensionful coupling constant, nonrenormalizable

A SECOND TRY

$$\mathcal{L}^{D=6} \; = \; \alpha \mathrm{Tr}\{F_{\mu\nu}\Box F_{\mu\nu}\} + \beta \mathrm{Tr}\{F_{\mu\nu}F_{\nu\alpha}F_{\alpha\mu}\}$$

- α, β are dimensionless, renormalizability
- Includes higher derivatives

But GHOSTS appear

DEFINITION

Ghost system is a system where the spectrum of the quantum Hamiltonian does not have a ground state.

• Ghosts are inherent for higher-derivative systems.

OSTROGRADSKY HAMILTONIAN

M. Ostrogradsky [1801-1862] is known by

- Ostrogradsky theorem from vector analysis
- Ostrogradsky method for calculating $\int P(x)/Q(x)\,dx$
- Ostrogradsky Hamiltonian

In the paper

[M. Ostrogradsky, *Mémoire sur les équations différentielles relatives au problème des isopérimètres*, Mem. Ac. St. Petersbourg **VI 4** (1850) 385.]

he reinvented the Hamiltonian formalism and applied it to higher-derivative theories.

- Consider $L(x, \dot{x}, \ddot{x})$.
- Equation of motion:

$$\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{\partial L}{\partial x} = 0\,.$$

- Conserved energy:

$$E = \ddot{x}\frac{\partial L}{\partial \ddot{x}} + \dot{x}\left(\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{x}}\right) - L\,.$$

- Treat $v = \dot{x}$ as an independent variable and define

$$p_v = \frac{\partial L}{\partial \dot{v}} = \frac{\partial L}{\partial \ddot{x}}\,,$$
$$p_x = \frac{\partial L}{\partial \dot{x}} - \dot{p}_v\,,$$

- Canonical Hamiltonian:

$$H(p_v, p_x; v, x) = p_v\dot{v} + p_x\dot{x} - L =$$
$$p_v a(p_v, x, v) + p_x v - L\left[a(p_v, x, v), x, v\right]\,,$$

where $a(p_v, x, v)$ is the solution of the equation $\partial L(x, v, a)/\partial a = p_v$.

- Linear term $p_x v \longrightarrow$ not positive definite.

• A. Pais and G.E. Uhlenbeck (1950) observed the presence of ghosts there.

GENERAL THEOREMS

Theorem 1: [R. Woodard, 2015] *The classical energy of a nondegenerate higher-derivative system can acquire an arbitrary positive or negative value.*

Theorem 2: [M. Raidal and H. Veermae, 2017] *The spectrum of a quantum Hamiltonian of a higher-derivative system is not bounded neither from below, nor from above.*

Example

$$H = \frac{\hat{P}_1^2 + \omega_1^2 X_1^2}{2} - \frac{\hat{P}_2^2 + \omega_2^2 X_2^2}{2} \,. \qquad (1)$$

The spectrum is

$$E_{nm} \;=\; \left(n + \frac{1}{2}\right)\omega_1 - \left(m + \frac{1}{2}\right)\omega_2$$

with positive integer n, m.

- All states are normalizable (“pure point”). Infinite degeneracy if ω_1/ω_2 is rational. Everywhere dense if ω_1/ω_2 is irrational.

UNUSUAL BUT NOT SICK!

PAIS-UHLENBECK OSCILLATOR:

$$L = \frac{1}{2}\left[\ddot{x}^2 - (\omega_1^2 + \omega_2^2)\dot{x}^2 + \omega_1^2\omega_2^2 x^2\right]$$

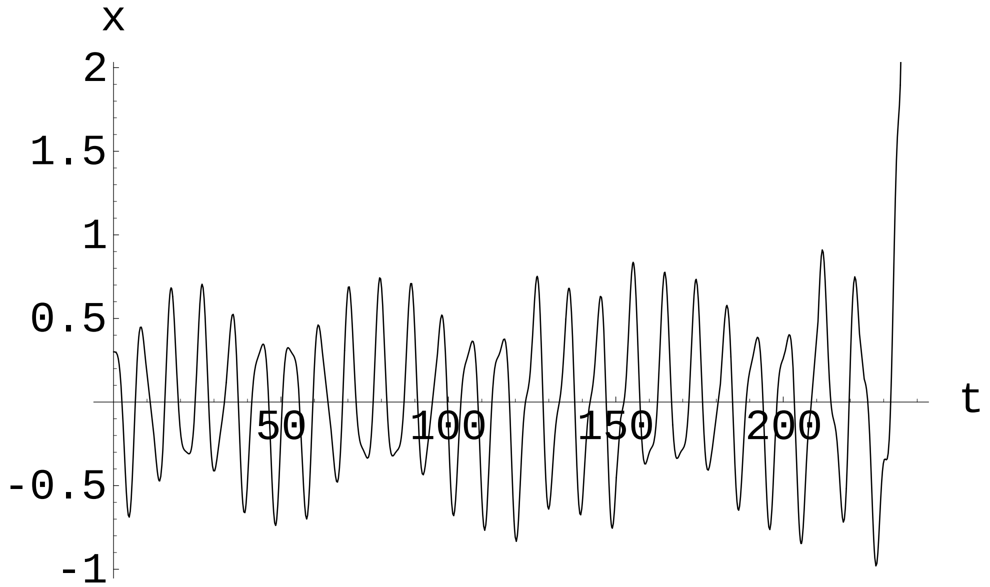
• If $\omega_1 \neq \omega_2$, its Ostrogradsky Hamiltonian is reduced to (1) by a canonical transformation [P.D. Mannheim and A.Davidson, 2005].

INCLUDING INTERACTIONS: DANGER OF COLLAPSE

• This happens for

$$L = \frac{1}{2}\left[\ddot{x}^2 - (\omega_1^2 + \omega_2^2)\dot{x}^2 + \omega_1^2\omega_2^2 x^2 - \lambda x^4\right]$$

ON THE SHORE OF THE STABILITY ISLAND



A similar stability island for another HD system in [S.N. Carrol, M. Hoffman, and M. Trodden, PR **D68** (2003) 023509]

Falling to the center

Consider

$$H = \frac{p^2}{2m} - \frac{\kappa}{r^2} \qquad (2)$$

Collapsing classical trajectories

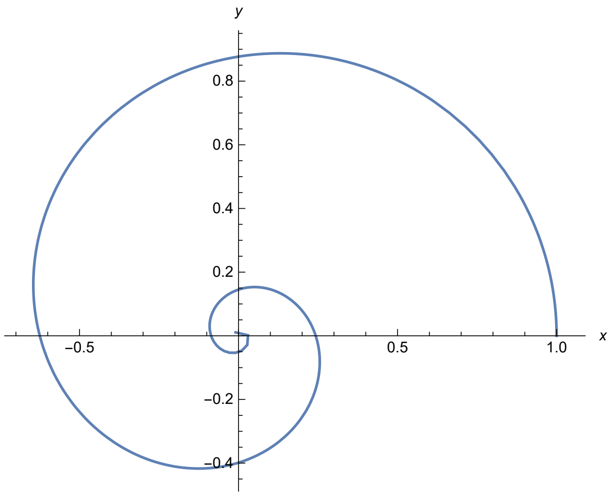


Figure 1: Falling on the center for the Hamiltonian (2) with $m = 1$ and $\kappa = .05$. The energy is slightly negative. The particles with positive energies escape to infinity.

- If $m\kappa > 1/8$, the quantum spectrum is not bounded from below.

• Schrödinger problem is not well defined.

If one smoothens the singularity,

$$V(r) = -\frac{\kappa}{r^2}, \qquad r > a\,,$$
$$V(r) = -\frac{\kappa}{a^2}, \qquad r \le a\,,$$

the spectrum is bounded, but depends on a.

• Violation of unitarity (probability "leaks" into the singularity). Ghosts are malignant here !

AN OBSERVATION:

• If quantum theory is sick, so is its classical counterpart. If classical theory is benign, so is its quantum counterpart.

THERE ARE INTERACTING SYSTEMS WITH BENIGN GHOSTS!

Example

[D. Robert and A.S., 2006]

$$H = pP - DV'(x)\,.$$

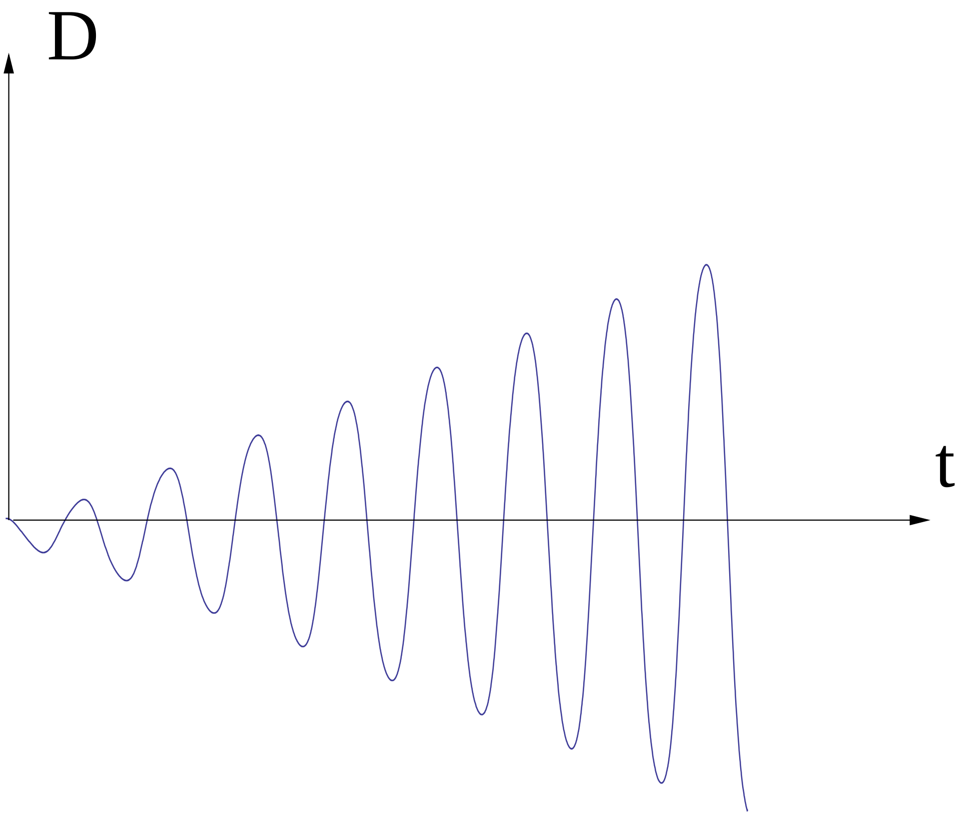
- This Hamiltonian is not positive definite
  - 4-dimensional phase space $(p, x), (P, D)$.
  - Two integrals of motion: H and

$$N = \frac{P^2}{2} + V(x)\,.$$

  - Exactly solvable.
  - Take

$$V = \frac{\omega^2 x^2}{2} + \frac{\lambda x^4}{4}$$

- The solutions to the classical equations of motion are expressed via elliptic functions.

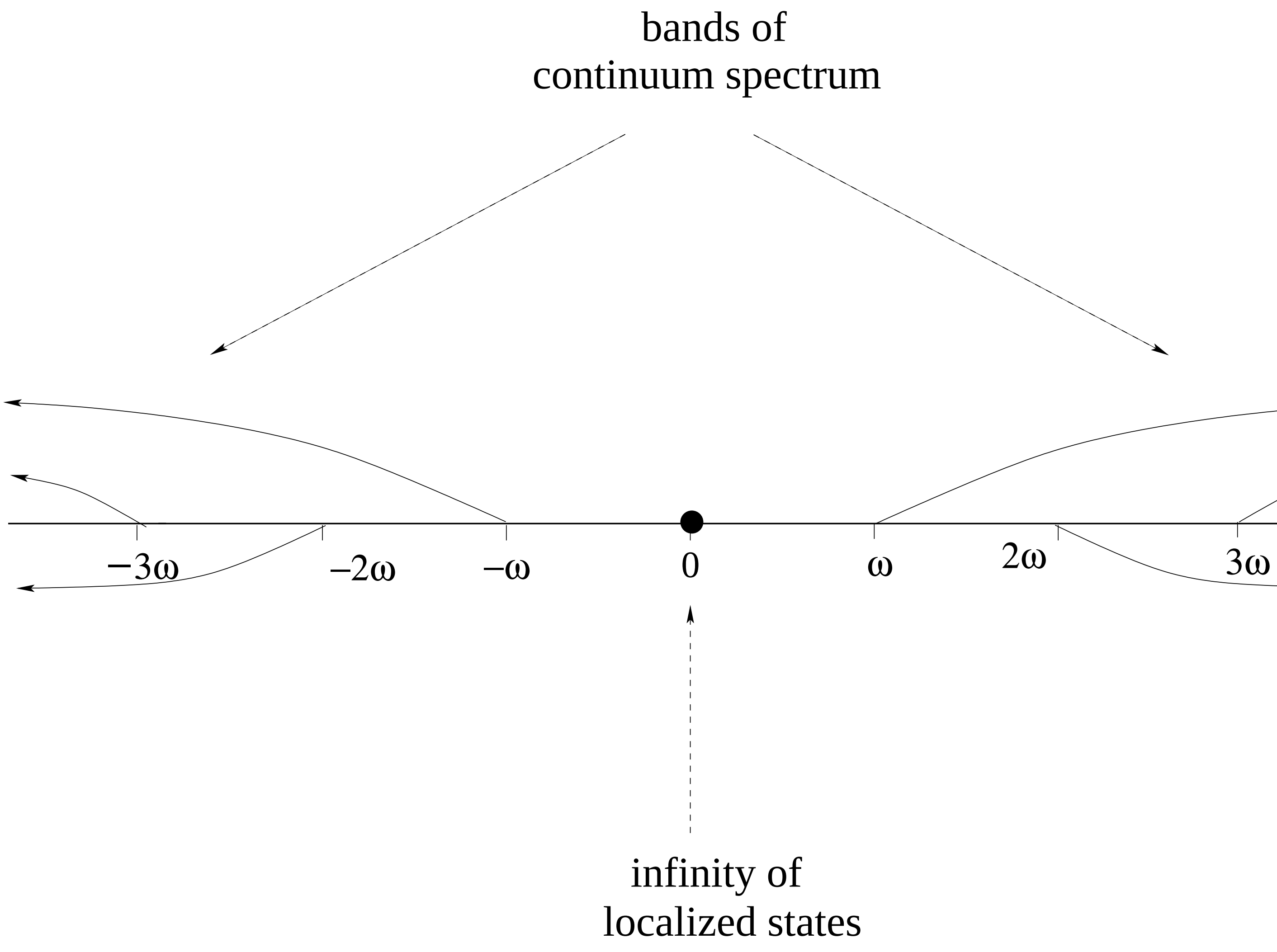


• Linear growth for $D(t)$; $x(t)$ is bounded. No blow up.

• Other benign ghost systems:
[M. Pavšič, 2013; I.B. Ilhan and A.Kovner, 2013; C. Deffayet, S. Mukohyama and A. Vikman, 2021]

QUANTUM PROBLEM

is also exactly solvable.



Spectrum of the Hamiltonian $H = pP - DV'(x)$.

Observation: INTEGRABILITY HELPS!

Example: Toda chain
[A.S., PLA, **389** (2021) 127104.]

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + V_{12} + V_{23} + V_{31} ,$$

where $V_{12} = e^{q_1 - q_2}$, etc. Besides the energy, the system involves an obvious integral of motion $P = p_1 + p_2 + p_3$, as well as the less obvious cubic invariant

$$I = \frac{1}{3}(p_1^3 + p_2^3 + p_3^3) + p_1(V_{12} + V_{31}) + p_2(V_{12} + V_{23}) + p_3(V_{23} + V_{31}) .$$

• Finite motion and discrete quantum spectrum.

THE MAIN IDEA: Treat I as a Hamiltonian!

• The eigenstates of $\hat{H}$ are also eigenstates of $\hat{I}$.

• The spectrum of $\hat{I}$ involves positive and negative eigenvalues and is not bounded from below!

Classical trajectories

(a)

(b)

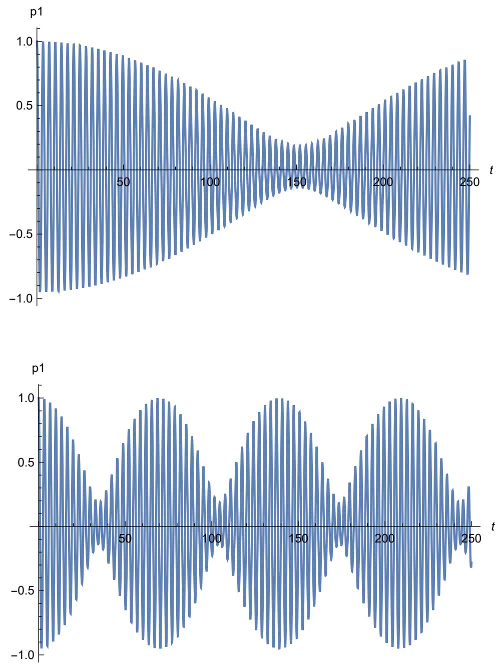


Figure 2: The dependence $p_1(t)$ for the equations of motion based on H and on I. The inital conditions are $q_1(0) = q_2(0) = q_3(0) = 0, p_1(0) = 1, p_2(0) = p_3(0) = -.5$.

NEW RESULTS

[T. Damour and A. Smilga, in preparation.]

I. Variation of an ordinary system.

• Take

$$L_0 = \frac{\dot{x}^2}{2} - V(x). \tag{3}$$

• Trade x for $x + \epsilon D$, expand on ϵ and keep only the linear term.

We obtain the Lagrangian

$$L_1(x, D, \dot{x}, \dot{D}) = \dot{x}\dot{D} - V'(x)D \tag{4}$$

with 2 pairs of dynamical variables.

• The classical energy of the system (3), $N = \dot{x}^2/2 + V(x)$, is still conserved. The Hamiltonian for (4), $H = pP + DV'(x)$, gives the second integral of motion.

• The trajectory $x(t)$ is the same as for (3). The solution is

$$x(t) = A_0(N) f[\Omega(N)(t - t_0)]$$

with a periodic $f(u)$.

• The trajectory $D(t)$ is a variation of $x(t)$. The variation of N gives linear growth in time. No blow-up.

Generalization

• Take some benign $L_0(q^i, \dot{q}^i)$. Replace $q^i \to q^i + \epsilon Q^i$, expand in ϵ and keep the linear term.

• The equations of motion for

$$L_1(q^i, \dot{q}^i; Q^i, \dot{Q}^i) \;=\; Q^i \frac{\partial L_0}{\partial q^j} + \dot{Q}^i \frac{\partial L_0}{\partial \dot{q}^j}$$

may admit only the linear growth in time. No blow-up. Hence no collapse and no violation of unitarity in the quantum problem.

• The same for field theories. Take e.g. the Yang-Mills Lagrangian

$$\mathcal{L} \;=\; -\frac{1}{2}\mathrm{Tr}\{F_{\mu\nu}F^{\mu\nu}\}\,, \tag{5}$$

set $A_\mu \to A_\mu + \epsilon B_\mu$, expand in ϵ and keep the linear terms. One obtains a nontrivial nonlinear field system with benign ghosts.

II. Geodesics on Lorentzian manifolds.

Example: De Sitter in 2 dimensions

$$ds^2 = (1+x^2)dt^2 - \frac{dx^2}{1+x^2}\,.$$

• The curvature $R = 2$ is constant.

Geodedic equations

$$\begin{aligned} \ddot{x} + x(x^2+1)\dot{t}^2 - \frac{x}{x^2+1}\dot{x}^2 &= 0\,, \\ \ddot{t} + \frac{2x}{x^2+1}\dot{t}\dot{x} &= 0 \end{aligned}$$

(with $\dot{x} \equiv dx/d\tau, \dot{t} \equiv dt/d\tau$, where τ is the proper time).

They follow from the ghost-ridden Hamiltonian

$$H = \frac{p_t^2}{2(1+x^2)} - \frac{1+x^2}{2}p_x^2\,.$$

• When the energy is positive, the trajectories are bounded.

• When the energy is negative, $x(\tau)$ grows exponentially, $x(\tau) \propto \sinh(\sqrt{-2E}\tau)$, but there is no blow-up!

• The geodesics on many other Lorentzian spaces have the same properties.

III. Modified KdV: a benign ghost-ridden field theory

- Consider the equation

$$u_t + 12u^2 u_x + u_{xxx} \;=\; 0\,. \tag{6}$$

- Replacing $x \leftrightarrow t$, we obtain an equation

$$u_x + 12u^2 u_t + u_{ttt} \;=\; 0 \tag{7}$$

including higher time derivatives. It follows from the Lagrangian

$$L \;=\; \frac{1}{2}\psi_{tt}^2 - \psi_t^4 - \frac{1}{2}\psi_t\psi_x\,, \tag{8}$$

where $u = \psi_t$. The corresponding Hamiltonian is not positive definite.

- We are interested in the classical temporal dynamics of the equation (7) $\equiv$ the dynamics of (6) in the spatial direction. The Cauchy problem for the latter consists in defining $u(t), u_x(t)$ and $u_{xx}(t)$ at some point $x = 0$.

- There are many analytical arguments (though no rigourous proof) that this dynamics is benign, no blow-up (not so for the ordinary KdV !).

ONE OF THE ARGUMENTS

Consider time-independent solutions. They satisfy the equation

$$u_{xxx} + 12u^2 u_x = \frac{\partial(u_{xx} + 4u^3)}{\partial x} = 0\,,$$

giving

$$u_{xx} + 4u^3 = C \tag{9}$$

This is a classical equation of motion for a particle in the potential

$$V(u) = u^4 - Cu\,. \tag{10}$$

The potential grows at infinity and the motion is finite.

- For the ordinary KdV, one would obtain the cubic potential and the corresponding equations of motion would have singular run-away solutions.

- Also for the equation

$$u_{xxx} - 12u^2 u_x + u_t = 0$$

we would have the negative quartic potential and run-away solutions.

NUMERICAL SIMULATIONS

• Consider Eq. (6) on the band $0 \leq t \leq 2\pi$ with the initial conditions: $u(t,0) = \sin t$, $u_x(t,0) = u_{xx}(t,0) = 0$. Mathematica gives:

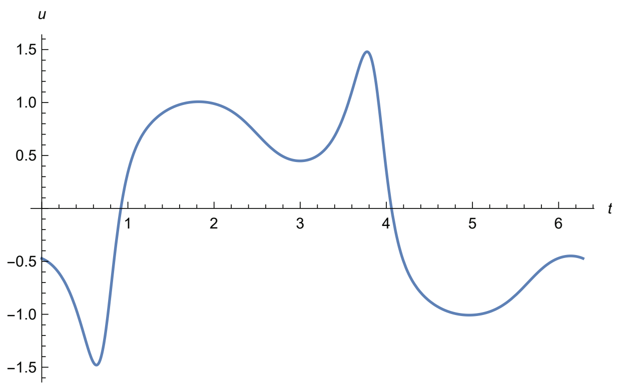


Figure 3: $u(t, x = 2.9)$

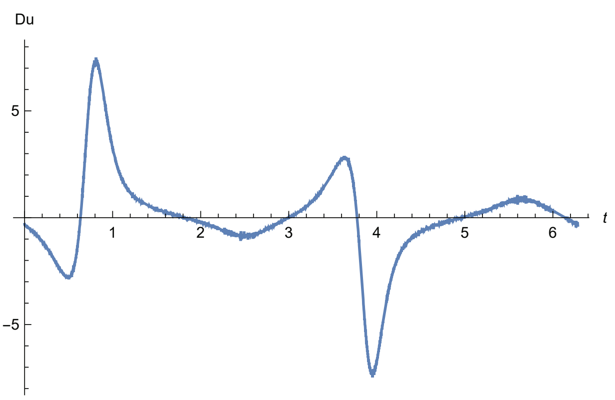


Figure 4: $u_x(t, x = 2.9)$

One sees a high-frequency numerical noise on the second plot. This noise does not allow to go much farther in x.

RELATED MECHANICAL SYSTEMS

One can expand $u(t,x)$ in a Fourrier series in t, take a finite number of modes and obtain out of (8) a mechanical Lagrangian with a finite number of degrees of freedom. The simplest discretized version of (8) reads

$$L = \frac{1}{2}(\psi - \chi)(\psi_x + \chi_x) + \frac{1}{2}\psi_{xx}^2 + \frac{1}{2}\chi_{xx}^2 - \psi_x^4 - \chi_x^4 \,.$$

It is higher-derivative involving only two dynamic variables $\psi(x)$ and $\chi(x)$, with x playing the role of time. We simulated the evolution of this system up to $x = 10000$ and have not found any blow-up.

THANK YOU
FOR ATTENTION!