Exactly solvable time-dependent non-Hermitian quantum systems from point transformations

Rebecca Tenney
City, University of London
Pseudo-Hermitian Hamiltonians in Quantum Physics

October 14, 2021

Exactly solvable time-dependent non-Hermitian quantum systems from point transformations

Rebecca Tenney
City, University of London
Pseudo-Hermitian Hamiltonians in Quantum Physics

October 14, 2021

Based on: A. Fring and R.Tenney, Phys. Lett. A. **410** 127548 (2021)

Outline

Outline

- Introduction

Outline

- Introduction
- Time-dependent non-Hermitian quantum systems
  - Key equations
  - The Dyson map, the metric
  - Different solution procedures

Outline

- Introduction
- Time-dependent non-Hermitian quantum systems
  - Key equations
  - The Dyson map, the metric
  - Different solution procedures
- Point transformations

Outline

- Introduction
- Time-dependent non-Hermitian quantum systems
    - Key equations
    - The Dyson map, the metric
    - Different solution procedures
- Point transformations
- Application to the Swanson Model

Outline

- Introduction
- Time-dependent non-Hermitian quantum systems
  - Key equations
  - The Dyson map, the metric
  - Different solution procedures
- Point transformations
- Application to the Swanson Model
- Conclusions

Introduction

Hamiltonians need not be Hermitian to have real eigenvalues.

Introduction

Hamiltonians need not be Hermitian to have real eigenvalues.
If a Hamiltonian is $\mathcal{PT}$-symmetric it has real eigenvalues.

$$\mathcal{PT}: \quad p \to p, \qquad x \to -x, \qquad i \to -i$$

Introduction

Hamiltonians need not be Hermitian to have real eigenvalues.
If a Hamiltonian is $\mathcal{PT}$-symmetric it has real eigenvalues.

$$\mathcal{PT}: \quad p \to p, \qquad x \to -x, \qquad i \to -i$$

$$H = p^2 + x^2 (ix)^\epsilon$$

$\varepsilon > 0 \to$ real eigenvalues

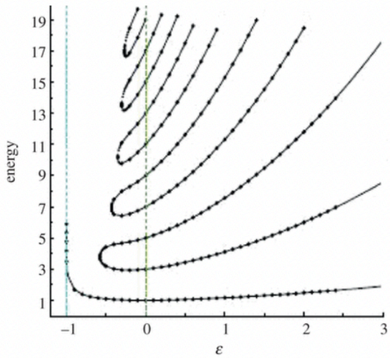

Figure 1: C. M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998)

Key equations

Two time-dependent Schrödinger equations for $h(t) = h^{\dagger}(t), H(t) \neq H^{\dagger}(t)$

$$h(t)\Psi(t) = i\hbar\partial_t\Psi(t) \qquad \text{and} \qquad H(t)\phi(t) = i\hbar\partial_t\phi(t)$$

Key equations

Two time-dependent Schrödinger equations for $h(t) = h^\dagger(t), H(t) \neq H^\dagger(t)$

$$h(t)\Psi(t) = i\hbar\partial_t\Psi(t) \qquad \text{and} \qquad H(t)\phi(t) = i\hbar\partial_t\phi(t)$$

Time-dependent Dyson map

$$\Psi(t) = \eta(t)\phi(t)$$

$\Longrightarrow$ Time-dependent Dyson equation (TDDE):

$$h(t) = \eta(t)H(t)\eta(t)^{-1} + i\hbar\partial_t\eta(t)\eta(t)^{-1}$$

Key equations

Two time-dependent Schrödinger equations for $h(t) = h^{\dagger}(t), H(t) \neq H^{\dagger}(t)$

$$h(t)\Psi(t) = i\hbar\partial_t\Psi(t) \qquad \text{and} \qquad H(t)\phi(t) = i\hbar\partial_t\phi(t)$$

Time-dependent Dyson map

$$\Psi(t) = \eta(t)\phi(t)$$

$\Longrightarrow$ Time-dependent Dyson equation (TDDE):

$$h(t) = \eta(t)H(t)\eta(t)^{-1} + i\hbar\partial_t\eta(t)\eta(t)^{-1}$$

$\Longrightarrow$ Time-dependent quasi-Hermiticity relation (TDQH):

$$H^{\dagger}(t)\rho(t) - \rho(t)H(t) = i\hbar\partial_t\rho(t), \quad \text{where} \quad \rho(t) = \eta^{\dagger}(t)\eta(t)$$

Key equations

Observables $o(t)$ in Hermitian system are self-adjoint.

Key equations

Observables $o(t)$ in Hermitian system are self-adjoint.
Observables $\mathcal{O}(t)$ in the non-Hermitian system are quasi-Hermitian.

Key equations

Observables $o(t)$ in Hermitian system are self-adjoint.
Observables $\mathcal{O}(t)$ in the non-Hermitian system are quasi-Hermitian.

$$o(t) = \eta(t)\mathcal{O}(t)\eta(t)^{-1}$$

Key equations

Observables $o(t)$ in Hermitian system are self-adjoint.
Observables $\mathcal{O}(t)$ in the non-Hermitian system are quasi-Hermitian.

$$o(t) = \eta(t)\mathcal{O}(t)\eta(t)^{-1}$$

Calculate observables:

$$\langle\Psi(t)|o(t)\Psi(t)\rangle = \langle\phi(t)|\rho(t)\mathcal{O}(t)\phi(t)\rangle$$

Key equations

Observables $o(t)$ in Hermitian system are self-adjoint.
Observables $\mathcal{O}(t)$ in the non-Hermitian system are quasi-Hermitian.

$$o(t) = \eta(t)\mathcal{O}(t)\eta(t)^{-1}$$

Calculate observables:

$$\langle\Psi(t)|o(t)\Psi(t)\rangle = \langle\phi(t)|\rho(t)\mathcal{O}(t)\phi(t)\rangle$$

$H(t)$ not observable $\rightarrow \tilde{H}(t) = \eta(t)^{-1}h(t)\eta(t)$ is observable energy operator.

Key equations

Observables $o(t)$ in Hermitian system are self-adjoint.
Observables $\mathcal{O}(t)$ in the non-Hermitian system are quasi-Hermitian.

$$o(t) = \eta(t)\mathcal{O}(t)\eta(t)^{-1}$$

Calculate observables:

$$\langle \Psi(t) | o(t)\Psi(t) \rangle = \langle \phi(t) | \rho(t)\mathcal{O}(t)\phi(t) \rangle$$

$H(t)$ not observable $\rightarrow \tilde{H}(t) = \eta(t)^{-1}h(t)\eta(t)$ is observable energy operator.
Must start by calculating $\rho(t)$ and $\eta(t)$!

How do we calculate $\rho(t)$ and $\eta(t)$?

[1] H. Lewis and W. Riesenfeld, J. Math. Phys. **10**, 1458-1473 (1969)

[2] B. Khantoul, A. Bounames and M. Maamache, The European Physical Journal Plus **132**(6), 258 (2017).

How do we calculate $\rho(t)$ and $\eta(t)$?

1. Solve TDDE directly for $\eta(t)$

[1] H. Lewis and W. Riesenfeld, J. Math. Phys. **10**, 1458-1473 (1969)
[2] B. Khantoul, A. Bounames and M. Maamache, The European Physical Journal Plus **132**(6), 258 (2017).

How do we calculate $\rho(t)$ and $\eta(t)$?

1. Solve TDDE directly for $\eta(t) \rightarrow \rho(t) = \eta^\dagger(t)\eta(t)$

[1] H. Lewis and W. Riesenfeld, J. Math. Phys. **10**, 1458-1473 (1969)

[2] B. Khantoul, A. Bounames and M. Maamache, The European Physical Journal Plus **132**(6), 258 (2017).

How do we calculate $\rho(t)$ and $\eta(t)$?

1. Solve TDDE directly for $\eta(t) \to \rho(t) = \eta^\dagger(t)\eta(t)$
2. Solve TDQH directly for $\rho(t)$

[1] H. Lewis and W. Riesenfeld, J. Math. Phys. **10**, 1458-1473 (1969)

[2] B. Khantoul, A. Bounames and M. Maamache, The European Physical Journal Plus **132**(6), 258 (2017).

How do we calculate $\rho(t)$ and $\eta(t)$?

1. Solve TDDE directly for $\eta(t) \rightarrow \rho(t) = \eta^{\dagger}(t)\eta(t)$
2. Solve TDQH directly for $\rho(t) \rightarrow$ harder to determine $\eta(t)$

[1] H. Lewis and W. Riesenfeld, J. Math. Phys. **10**, 1458-1473 (1969)

[2] B. Khantoul, A. Bounames and M. Maamache, The European Physical Journal Plus **132**(6), 258 (2017).

How do we calculate $\rho(t)$ and $\eta(t)$?

1. Solve TDDE directly for $\eta(t) \rightarrow \rho(t) = \eta^{\dagger}(t)\eta(t)$
2. Solve TDQH directly for $\rho(t) \rightarrow$ harder to determine $\eta(t)$
3. Lewis-Riesenfeld invariants[1] :

$$\frac{dI_{\mathcal{H}}(t)}{dt} = \partial_t I_{\mathcal{H}}(t) - i\hbar\left[I_{\mathcal{H}}(t), \mathcal{H}(t)\right] = 0, \text{for} \quad \mathcal{H} = h = h^{\dagger}, H \neq H^{\dagger}$$

[1] H. Lewis and W. Riesenfeld, J. Math. Phys. **10**, 1458-1473 (1969)

[2] B. Khantoul, A. Bounames and M. Maamache, The European Physical Journal Plus **132**(6), 258 (2017).

How do we calculate $\rho(t)$ and $\eta(t)$?

1. Solve TDDE directly for $\eta(t) \rightarrow \rho(t) = \eta^\dagger(t)\eta(t)$
2. Solve TDQH directly for $\rho(t) \rightarrow$ harder to determine $\eta(t)$
3. Lewis-Riesenfeld invariants[1] :

$$\frac{dI_{\mathcal{H}}(t)}{dt} = \partial_t I_{\mathcal{H}}(t) - i\hbar \left[I_{\mathcal{H}}(t), \mathcal{H}(t) \right] = 0, \text{for} \quad \mathcal{H} = h = h^\dagger, H \neq H^\dagger$$

Invariants are quasi-Hermitian[2]

$$I_h(t) = \eta(t) I_H \eta(t)^{-1}$$

[1] H. Lewis and W. Riesenfeld, J. Math. Phys. **10**, 1458-1473 (1969)

[2] B. Khantoul, A. Bounames and M. Maamache, The European Physical Journal Plus **132**(6), 258 (2017).

How do we calculate $\rho(t)$ and $\eta(t)$?

1. Solve TDDE directly for $\eta(t) \to \rho(t) = \eta^\dagger(t)\eta(t)$
2. Solve TDQH directly for $\rho(t) \to$ harder to determine $\eta(t)$
3. Lewis-Riesenfeld invariants[1] :

$$\frac{dI_{\mathcal{H}}(t)}{dt} = \partial_t I_{\mathcal{H}}(t) - i\hbar\left[I_{\mathcal{H}}(t), \mathcal{H}(t)\right] = 0, \text{for} \quad \mathcal{H} = h = h^\dagger, H \neq H^\dagger$$

Invariants are quasi-Hermitian[2]

$$I_h(t) = \eta(t) I_H \eta(t)^{-1}$$

Solution to TDSE:

$$I_{\mathcal{H}} \left|\phi_{\mathcal{H}}(t)\right\rangle = \lambda \left|\phi_{\mathcal{H}}(t)\right\rangle, \quad \left|\Psi_{\mathcal{H}}(t)\right\rangle = e^{i\hbar\alpha(t)} \left|\phi_{\mathcal{H}}(t)\right\rangle$$

$$\dot{\alpha}(t) = \left\langle\phi_{\mathcal{H}}(t)\right| (i\hbar\partial_t - \mathcal{H}) \left|\phi_{\mathcal{H}}(t)\right\rangle, \qquad \dot{\lambda} = 0$$

[1] H. Lewis and W. Riesenfeld, J. Math. Phys. **10**, 1458-1473 (1969)

[2] B. Khantoul, A. Bounames and M. Maamache, The European Physical Journal Plus **132**(6), 258 (2017).

Four step approach

For information on point transformations constructed between Hermitian Hamiltonians see[3]

- Two time-dependent Schrödinger equations:

$$H_0(\chi)\psi(\chi,\tau) = i\hbar\partial_\tau\psi(\chi,\tau) \qquad \text{and} \qquad H(x,t)\phi(x,t) = i\hbar\partial_t\phi(x,t)$$

Reference Hamiltonian: $H_0(\chi)$ Target Hamiltonian: $H(x,t) \neq H^\dagger(x,t)$

[3] K. Zelaya and O. Rosas-Ortiz, Physica Scripta **95**(6), 064004 (2020).

Four step approach

For information on point transformations constructed between Hermitian Hamiltonians see[3]

- Two time-dependent Schrödinger equations:

$$H_0(\chi)\psi(\chi,\tau) = i\hbar\partial_\tau\psi(\chi,\tau) \qquad \text{and} \qquad H(x,t)\phi(x,t) = i\hbar\partial_t\phi(x,t)$$

Reference Hamiltonian: $H_0(\chi)$ Target Hamiltonian: $H(x,t) \neq H^\dagger(x,t)$

- Point transformation Γ:

$$\Gamma : H_0 - \text{TDSE} \to H - \text{TDSE} \qquad [\chi,\tau,\psi(\chi,\tau)] \to [x,t,\phi(x,t)]$$

$$\chi = P(x,t,\phi) \quad \tau = Q(x,t,\phi) \quad \psi = R(x,t,\phi)$$

[3] K. Zelaya and O. Rosas-Ortiz, Physica Scripta **95**(6), 064004 (2020).

Four step approach

- Construction of invariant

$$\Gamma : H_0(\chi) \to I_H(x,t)$$

Four step approach

- Construction of invariant

$$\Gamma : H_0(\chi) \to I_H(x,t)$$

$$\frac{dI_H(x,t)}{dt} = i\hbar\partial_t I_H(x,t) + [I_H(x,t), H(x,t)] = 0$$

Four step approach

- Construction of invariant

$$\Gamma : H_0(\chi) \to I_H(x,t)$$

$$\frac{dI_H(x,t)}{dt} = i\hbar\partial_t I_H(x,t) + [I_H(x,t), H(x,t)] = 0$$

- Determine $\eta(t)$ and $\rho(t)$:

$$I_h(t) = \eta(t) I_H(x,t) \eta(t)^{-1}$$

Four step approach

- Construction of invariant

$$\Gamma : H_0(\chi) \to I_H(x,t)$$

$$\frac{dI_H(x,t)}{dt} = i\hbar\partial_t I_H(x,t) + [I_H(x,t), H(x,t)] = 0$$

- Determine $\eta(t)$ and $\rho(t)$:

$$I_h(t) = \eta(t) I_H(x,t) \eta(t)^{-1}$$

$$\implies h(t) = \eta(t) H(x,t) \eta^{-1} + i\hbar\partial_t \eta(t) \eta(t)^{-1}$$

$$\implies \rho(t) = \eta^{\dagger}(t)\eta(t)$$

The reference Hamiltonian

The reference Hamiltonian

Time-independent harmonic oscillator:

$$H_0(\chi) = \frac{P^2}{2m} + \frac{1}{2}m\omega^2\chi^2, \qquad P = -i\hbar\partial_\chi$$

The reference Hamiltonian

Time-independent harmonic oscillator:

$$H_0(\chi) = \frac{P^2}{2m} + \frac{1}{2}m\omega^2\chi^2, \qquad P = -i\hbar\partial_\chi$$

$$\chi = \chi(x,t), \qquad \tau(t), \qquad \psi = A(x,t)\phi,$$

The reference Hamiltonian

Time-independent harmonic oscillator:

$$H_0(\chi) = \frac{P^2}{2m} + \frac{1}{2} m\omega^2 \chi^2, \qquad P = -i\hbar\partial_\chi$$

$$\chi = \chi(x,t), \qquad \tau(t), \qquad \psi = A(x,t)\phi,$$

- Simplify the calculation.

The reference Hamiltonian

Time-independent harmonic oscillator:

$$H_0(\chi) = \frac{P^2}{2m} + \frac{1}{2}m\omega^2\chi^2, \qquad P = -i\hbar\partial_\chi$$

$$\chi = \chi(x,t), \qquad \tau(t), \qquad \psi = A(x,t)\phi,$$

- Simplify the calculation.
- No ϕ_x^2 term so require $\psi_{\phi\phi} = 0$.

The Reference Hamiltonian

Compute the total derivatives:

$$\frac{d\psi}{dx} = \psi_\chi \chi_x = A\phi_x + A_x\phi$$
$$\frac{d\psi}{dt} = \psi_\chi \chi_t + \psi_\tau \tau_t = A\phi_t + A_t\phi$$
$$\frac{d^2\psi}{dx^2} = \psi_{\chi,\chi}\chi_x^2 + \psi_\chi \chi_{x,x} = A\phi_{x,x}2A_x\phi_x + \psi A_{x,x}$$

The Reference Hamiltonian

Compute the total derivatives:

$$\frac{d\psi}{dx} = \psi_\chi \chi_x = A\phi_x + A_x\phi$$

$$\frac{d\psi}{dt} = \psi_\chi \chi_t + \psi_\tau \tau_t = A\phi_t + A_t\phi$$

$$\frac{d^2\psi}{dx^2} = \psi_{\chi,\chi}\chi_x^2 + \psi_\chi \chi_{x,x} = A\phi_{x,x} 2A_x\phi_x + \psi A_{x,x}$$

$\rightarrow$ Solve for ψ_χ, ψ_τ and $\psi_{\chi,\chi}$ and sub into TDSE for $H_0(\chi)$

The Reference Hamiltonian

Point transformed TDSE:

$$i\hbar\phi_t + \frac{\hbar^2}{2m}\frac{\tau_t}{\chi_x^2}\phi_{xx} + B_0(x,t)\phi_x - V_0(x,t)\phi = 0 \qquad (*)$$

where

$$B_0(x,t) = -i\hbar\frac{\chi_t}{\chi_x} + \frac{\hbar^2}{2m}\frac{\tau_t}{\chi_x^2}\left(2\frac{A_x}{A} - \frac{\chi_{xx}}{\chi_x}\right)$$

$$V_0(x,t) = \frac{1}{2}m\omega^2\tau_t\chi^2 - i\hbar\left(\frac{A_t}{A} - \frac{A_x\chi_t}{A\chi_x}\right) - \frac{\hbar^2}{2m}\frac{\tau_t}{\chi_x^2}\left(\frac{A_{xx}}{A} - \frac{A_x\chi_{xx}}{A\chi_x}\right)$$

The Reference Hamiltonian

Point transformed TDSE:

$$i\hbar\phi_t + \frac{\hbar^2}{2m}\frac{\tau_t}{\chi_x^2}\phi_{xx} + B_0(x,t)\phi_x - V_0(x,t)\phi = 0 \qquad (*)$$

where

$$B_0(x,t) = -i\hbar\frac{\chi_t}{\chi_x} + \frac{\hbar^2}{2m}\frac{\tau_t}{\chi_x^2}\left(2\frac{A_x}{A} - \frac{\chi_{xx}}{\chi_x}\right)$$

$$V_0(x,t) = \frac{1}{2}m\omega^2\tau_t\chi^2 - i\hbar\left(\frac{A_t}{A} - \frac{A_x\chi_t}{A\chi_x}\right) - \frac{\hbar^2}{2m}\frac{\tau_t}{\chi_x^2}\left(\frac{A_{xx}}{A} - \frac{A_x\chi_{xx}}{A\chi_x}\right)$$

$\rightarrow$ Compare (*) directly with TDSE for target Hamiltonian $H(x,t)$ and solve for A, χ and τ.

The reference Hamiltonian - other choices

$$i\hbar\phi_t + \frac{\hbar^2}{2m}\frac{\tau_t}{\chi_x^2}\phi_{xx} + B_i(x,t)\phi_x - V_i(x,t)\phi = 0$$

The reference Hamiltonian - other choices

$$i\hbar\phi_t + \frac{\hbar^2}{2m}\frac{\tau_t}{\chi_x^2}\phi_{xx} + B_i(x,t)\phi_x - V_i(x,t)\phi = 0$$

$$H_0^{(1)}(\chi) = \frac{P^2}{2m}, \qquad B_1(x,t) = B_0(x,t), \qquad V_1(x,t) = V_0(x,t) - \frac{1}{2}m\omega^2\chi^2\tau_t$$

The reference Hamiltonian - other choices

$$i\hbar\phi_t + \frac{\hbar^2}{2m}\frac{\tau_t}{\chi_x^2}\phi_{xx} + B_i(x,t)\phi_x - V_i(x,t)\phi = 0$$

$$H_0^{(1)}(\chi) = \frac{P^2}{2m}, \qquad B_1(x,t) = B_0(x,t), \qquad V_1(x,t) = V_0(x,t) - \frac{1}{2}m\omega^2\chi^2\tau_t$$

$$H_0^{(2)}(\chi) = H_0(\chi) + a\chi, \qquad B_2(x,t) = B_0(x,t), \qquad V_2(x,t) = V_0(x,t) + a\chi\tau_t$$

The reference Hamiltonian - other choices

$$i\hbar\phi_t + \frac{\hbar^2}{2m}\frac{\tau_t}{\chi_x^2}\phi_{xx} + B_i(x,t)\phi_x - V_i(x,t)\phi = 0$$

$$H_0^{(1)}(\chi) = \frac{P^2}{2m}, \qquad B_1(x,t) = B_0(x,t), \qquad V_1(x,t) = V_0(x,t) - \frac{1}{2}m\omega^2\chi^2\tau_t$$

$$H_0^{(2)}(\chi) = H_0(\chi) + a\chi, \qquad B_2(x,t) = B_0(x,t), \qquad V_2(x,t) = V_0(x,t) + a\chi\tau_t$$

$$H_0^{(3)}(\chi) = H_0(\chi) + a\{\chi, P\}, \quad B_3(x,t) = B_0(x,t) + \frac{2ia\hbar\chi\tau_t}{\chi_x}, \quad V_3(x,t) = V_0(x,t) - \frac{2ia\chi_x\tau_t}{A\chi_x} - ia\hbar\tau_t$$

The time-dependent Swanson model

The time-dependent Swanson model

$$\tilde{H}_S(t) = \omega(t)\left(a^\dagger a + 1/2\right) + \tilde{\alpha}(t)a^2 + \tilde{\beta}(t)\left(a^\dagger\right)^2, \qquad \tilde{\alpha} \neq \tilde{\beta}^*$$

$$a = (x + ip)/2, \qquad a^\dagger = (x - ip)/2$$

The time-dependent Swanson model

The time-dependent Swanson model

$$\tilde{H}_S(t) = \omega(t)\left(a^\dagger a + 1/2\right) + \tilde{\alpha}(t) a^2 + \tilde{\beta}(t)\left(a^\dagger\right)^2, \qquad \tilde{\alpha} \neq \tilde{\beta}^*$$

$$a = (x + ip)/2, \qquad a^\dagger = (x - ip)/2$$

$$\tilde{\alpha} = \frac{M\Omega^2}{4} - \frac{1}{4M} + \alpha,\ \tilde{\beta} = \frac{M\Omega^2}{4} - \frac{1}{4M} - \alpha,\ \omega = \frac{M\Omega^2}{2} + \frac{1}{2M},$$

$$H_S(x,t) := \tilde{H}_S(t) - \frac{\omega(t)}{2} = \frac{p^2}{2M(t)} + \frac{M(t)}{2}\Omega(t)^2 x^2 + i\alpha(t)\{x,p\},\ \ M,\Omega \in \mathbb{R},\ \ \alpha \in \mathbb{C}$$

The time-dependent Swanson model

The time-dependent Swanson model

$$\tilde{H}_S(t) = \omega(t)\left(a^\dagger a + 1/2\right) + \tilde{\alpha}(t)a^2 + \tilde{\beta}(t)\left(a^\dagger\right)^2, \qquad \tilde{\alpha} \neq \tilde{\beta}^*$$

$$a = (x + ip)/2, \qquad a^\dagger = (x - ip)/2$$

$$\tilde{\alpha} = \frac{M\Omega^2}{4} - \frac{1}{4M} + \alpha, \, \tilde{\beta} = \frac{M\Omega^2}{4} - \frac{1}{4M} - \alpha, \, \omega = \frac{M\Omega^2}{2} + \frac{1}{2M},$$

$$H_S(x,t) := \tilde{H}_S(t) - \frac{\omega(t)}{2} = \frac{p^2}{2M(t)} + \frac{M(t)}{2}\Omega(t)^2 x^2 + i\alpha(t)\{x,p\}, \;\; M,\Omega \in \mathbb{R}, \;\; \alpha \in \mathbb{C}$$

$$\mathcal{PT}: x \to -x, \;\; p \to p, \;\; i \to -i, \;\; (M,\Omega,\alpha) \to (M,\Omega,\alpha), \quad \alpha = \alpha_r + i\alpha_i$$

$$\alpha_r \to \alpha_r, \;\; \alpha_i \to -\alpha_i$$

Point transformation $\Gamma_0^S : H_0(\chi) \to H_s(x,t)$

Point transformed TDSE:

$$i\hbar\psi_t + \frac{\hbar^2}{2m}\frac{\tau_t}{\chi_x^2}\phi_{xx} + B_0(x,t)\phi_x - V_0(x,t)\phi = 0 \quad (*)$$

TDSE for $H_s(x,t)$:

$$i\hbar\phi_t + \frac{\hbar^2}{2M(t)}\phi_{xx} - 2\hbar\alpha(t)x\phi_x - \hbar\alpha(t)\phi - \frac{1}{2}M(t)\Omega(t)^2x^2\phi = 0, \quad (\circ)$$

Point transformation $\Gamma_0^S : H_0(\chi) \to H_s(x,t)$

Point transformed TDSE:

$$i\hbar\psi_t + \frac{\hbar^2}{2m}\frac{\tau_t}{\chi_x^2}\phi_{xx} + B_0(x,t)\phi_x - V_0(x,t)\phi = 0 \quad (*)$$

TDSE for $H_s(x,t)$:

$$i\hbar\phi_t + \frac{\hbar^2}{2M(t)}\phi_{xx} - 2\hbar\alpha(t)x\phi_x - \hbar\alpha(t)\phi - \frac{1}{2}M(t)\Omega(t)^2x^2\phi = 0, \quad (\circ)$$

Compare $(*)$ and $(\circ)$:

$$\frac{\tau_t}{m\chi_x^2} = \frac{1}{M(t)},$$

Point transformation $\Gamma_0^S : H_0(\chi) \to H_s(x,t)$

Point transformed TDSE:

$$i\hbar\psi_t + \frac{\hbar^2}{2m}\frac{\tau_t}{\chi_x^2}\phi_{xx} + B_0(x,t)\phi_x - V_0(x,t)\phi = 0 \quad (*)$$

TDSE for $H_s(x,t)$:

$$i\hbar\phi_t + \frac{\hbar^2}{2M(t)}\phi_{xx} - 2\hbar\alpha(t)x\phi_x - \hbar\alpha(t)\phi - \frac{1}{2}M(t)\Omega(t)^2x^2\phi = 0, \quad (\circ)$$

Compare $(*)$ and $(\circ)$:

$$\frac{\tau_t}{m\chi_x^2} = \frac{1}{M(t)}, \quad B(x,t) = -2\hbar\alpha(t)x,$$

Point transformation $\Gamma_0^S : H_0(\chi) \to H_s(x,t)$

Point transformed TDSE:

$$i\hbar\psi_t + \frac{\hbar^2}{2m}\frac{\tau_t}{\chi_x^2}\phi_{xx} + B_0(x,t)\phi_x - V_0(x,t)\phi = 0 \quad (*)$$

TDSE for $H_s(x,t)$:

$$i\hbar\phi_t + \frac{\hbar^2}{2M(t)}\phi_{xx} - 2\hbar\alpha(t)x\phi_x - \hbar\alpha(t)\phi - \frac{1}{2}M(t)\Omega(t)^2x^2\phi = 0, \quad (\circ)$$

Compare $(*)$ and $(\circ)$:

$$\frac{\tau_t}{m\chi_x^2} = \frac{1}{M(t)}, \quad B(x,t) = -2\hbar\alpha(t)x, \quad V(x,t) = \frac{1}{2}M(t)\Omega(t)x^2 + \hbar\alpha(t)$$

Point transformation $\Gamma_0^S : H_0(\chi) \to H_s(x,t)$

Solution:

$$M(t) = m\sigma^{-r-2s}, \quad \tau(t) = \int^t \sigma(y)^r dy, \quad \chi(x,t) = \frac{x+\gamma(t)}{\sigma(t)^s}$$

Point transformation $\Gamma_0^S : H_0(\chi) \to H_s(x,t)$

Solution:

$$M(t) = m\sigma^{-r-2s}, \quad \tau(t) = \int^t \sigma(y)^r dy, \quad \chi(x,t) = \frac{x+\gamma(t)}{\sigma(t)^s}$$

$\gamma = 0$:

$$A(x,t) = \exp\left\{ \frac{im\sigma^{-1-r-2s}}{\hbar} \left[\left(i\alpha\sigma - \frac{1}{2} s\sigma_t \right) x^2 + \delta(t) \right] \right\}$$

$$\delta(t) = \sigma^{1+r+2s} \left(c_1 - \frac{is\hbar}{2m} \log\sigma \right)$$

Point transformation $\Gamma_0^S : H_0(\chi) \to H_s(x,t)$

Solution:

$$M(t) = m\sigma^{-r-2s}, \quad \tau(t) = \int^t \sigma(y)^r dy, \quad \chi(x,t) = \frac{x+\gamma(t)}{\sigma(t)^s}$$

$\gamma = 0$:

$$A(x,t) = \exp\left\{ \frac{im\sigma^{-1-r-2s}}{\hbar} \left[\left(i\alpha\sigma - \frac{1}{2}s\sigma_t \right) x^2 + \delta(t) \right] \right\}$$

$$\delta(t) = \sigma^{1+r+2s} \left(c_1 - \frac{is\hbar}{2m} \log\sigma \right)$$

$\mathcal{PT}$-symmetry:

$$\alpha_i = \frac{1}{4}\partial_t \ln\left(\frac{\sigma^{r+2s}}{\alpha_r} \right)$$

Point transformation $\Gamma_0^S : H_0(\chi) \to H_s(x,t)$

Solution:

$$M(t) = m\sigma^{-r-2s}, \quad \tau(t) = \int^t \sigma(y)^r dy, \quad \chi(x,t) = \frac{x+\gamma(t)}{\sigma(t)^s}$$

$\gamma = 0$:

$$A(x,t) = \exp\left\{ \frac{im\sigma^{-1-r-2s}}{\hbar} \left[\left(i\alpha\sigma - \frac{1}{2}s\sigma_t \right) x^2 + \delta(t) \right] \right\}$$

$$\delta(t) = \sigma^{1+r+2s} \left(c_1 - \frac{is\hbar}{2m} \log\sigma \right)$$

$\mathcal{PT}$-symmetry:

$$\alpha_i = \frac{1}{4}\partial_t \ln\left(\frac{\sigma^{r+2s}}{\alpha_r} \right) \quad \rightarrow \quad \alpha_i \propto \partial_t$$

Point transformation $\Gamma_0^S : H_0(\chi) \to H_s(x,t)$

Solution:

$$M(t) = m\sigma^{-r-2s}, \quad \tau(t) = \int^t \sigma(y)^r dy, \quad \chi(x,t) = \frac{x+\gamma(t)}{\sigma(t)^s}$$

$\gamma = 0$:

$$A(x,t) = \exp\left\{ \frac{im\sigma^{-1-r-2s}}{\hbar} \left[\left(i\alpha\sigma - \frac{1}{2} s\sigma_t \right) x^2 + \delta(t) \right] \right\}$$

$$\delta(t) = \sigma^{1+r+2s} \left(c_1 - \frac{is\hbar}{2m} \log\sigma \right)$$

$\mathcal{PT}$-symmetry:

$$\alpha_i = \frac{1}{4}\partial_t \ln\left(\frac{\sigma^{r+2s}}{\alpha_r}\right) \quad \to \quad \alpha_i \propto \partial_t \quad \to \quad \mathcal{PT} : \alpha_i \to -\alpha_i$$

Point transformation $\Gamma_0^S : H_0(\chi) \rightarrow H_s(x,t)$

Auxiliary equation:

$$\sigma_{tt} = \sigma \left[\frac{2\alpha_r(2\Omega^2\alpha_r + 8\alpha_r^3 + (\alpha_r)_{tt}) - 3(\alpha_r)_t^2}{2r\alpha_r^2} \right] + \frac{(\frac{r}{2}+1)\sigma_t^2}{\sigma} - \frac{2\omega^2\sigma^{2r+1}}{r}$$

[4] V. Ermakov, Univ. Izv. Kiev. **20**, 1-19 (1880).

[5] E. Pinney, Proc. Amer. Math. Soc **1**, 681(1) (1950).

Point transformation $\Gamma_0^S : H_0(\chi) \to H_s(x,t)$

Auxiliary equation:

$$\sigma_{tt} = \sigma\left[\frac{2\alpha_r(2\Omega^2\alpha_r + 8\alpha_r^3 + (\alpha_r)_{tt}) - 3(\alpha_r)_t^2}{2r\alpha_r^2}\right] + \frac{(\frac{r}{2}+1)\sigma_t^2}{\sigma} - \frac{2\omega^2\sigma^{2r+1}}{r}$$

Many choices for r, s and α_r, for example:
Time-independent mass:

- $\alpha_r = c_2\sigma^{r+2s}$, $r=-2s$, $s=1 \to \alpha_i = 0, \alpha$ is time-independent, $\sigma_{tt} = -c_2^2\sigma + \frac{\omega^2}{\sigma^3} - \sigma\Omega^2$

[4] V. Ermakov, Univ. Izv. Kiev. **20**, 1-19 (1880).
[5] E. Pinney, Proc. Amer. Math. Soc **1**, 681(1) (1950).

Point transformation $\Gamma_0^S : H_0(\chi) \to H_s(x,t)$

Auxiliary equation:

$$\sigma_{tt} = \sigma\left[\frac{2\alpha_r(2\Omega^2\alpha_r + 8\alpha_r^3 + (\alpha_r)_{tt}) - 3(\alpha_r)_t^2}{2r\alpha_r^2}\right] + \frac{(\frac{r}{2}+1)\sigma_t^2}{\sigma} - \frac{2\omega^2\sigma^{2r+1}}{r}$$

Many choices for r, s and α_r, for example:
Time-independent mass:

- $\alpha_r = c_2\sigma^{r+2s}$, $r = -2s$, $s = 1 \to \alpha_i = 0$, α is time-independent,
  $\sigma_{tt} = -c_2^2\sigma + \frac{\omega^2}{\sigma^3} - \sigma\Omega^2$

α complex

- $\alpha_r = \sigma^{-2-r}$, $r = 0$
  $\sigma_{tt} = \frac{4}{\sigma^3} + \sigma(\Omega^2 - \omega^2)$

σ : non-linear Ermakov-Pinney equation[4,5]

[4] V. Ermakov, Univ. Izv. Kiev. **20**, 1-19 (1880).
[5] E. Pinney, Proc. Amer. Math. Soc **1**, 681(1) (1950).

Non-Hermitian invariant

Construct non-Hermitian invariant from the point transformation:

$$\Gamma_0^S : H_0(\chi) \rightarrow I_H(x,t)$$

Non-Hermitian invariant

Construct non-Hermitian invariant from the point transformation:

$$\Gamma_0^S : H_0(\chi) \to I_H(x,t)$$

$$\hat{I}_H(x,t) = \frac{\sigma^{2s}}{2m}p^2 + \frac{\sigma^{-r-1}\left(4i\sigma\alpha_r^2 + r\alpha_r\sigma_t - \sigma\alpha_{rt}\right)}{4\alpha_r}\{x,p\} + \frac{\sigma^{-2(r+s+1)}\left[4m\omega^2\alpha_r^2\sigma^{2r+2} - m\left(4\sigma\alpha_r^2 - ir\alpha_r\sigma_t + i\sigma\alpha_{rt}\right)^2\right]}{8\alpha_r^2}x^2$$

Non-Hermitian invariant

Construct non-Hermitian invariant from the point transformation:

$$\Gamma_0^S : H_0(\chi) \to I_H(x,t)$$

$$\hat{I}_H(x,t) = \frac{\sigma^{2s}}{2m} p^2 + \frac{\sigma^{-r-1}\left(4i\sigma\alpha_r^2 + r\alpha_r\sigma_t - \sigma\alpha_{rt}\right)}{4\alpha_r}\{x,p\} + \frac{\sigma^{-2(r+s+1)}\left[4m\omega^2\alpha_r^2\sigma^{2r+2} - m\left(4\sigma\alpha_r^2 - ir\alpha_r\sigma_t + i\sigma\alpha_{rt}\right)^2\right]}{8\alpha_r^2} x^2$$

Different reference Hamiltonians $\to$ different invariants.

The Dyson map and metric

All invariants can be written as ($\gamma \neq 0$)

$$I_H = a_r p^2 + b_r p + (c_r + ic_i)\{x, p\} + (d_r + id_i)\,x^2 + (e_r + ie_i)\,x + f_r,$$

The Dyson map and metric

All invariants can be written as ($\gamma \neq 0$)

$$I_H = a_r p^2 + b_r p + (c_r + ic_i)\{x, p\} + (d_r + id_i)\, x^2 + (e_r + ie_i)\, x + f_r,$$

Property:

$$\frac{e_i}{2b_r} = \frac{d_i}{4c_r} = \frac{c_i}{2a_r} = m\alpha_r \sigma^{-r-2s}.$$

The Dyson map and metric

All invariants can be written as ($\gamma \neq 0$)

$$I_H = a_r p^2 + b_r p + (c_r + ic_i)\{x, p\} + (d_r + id_i)\,x^2 + (e_r + ie_i)\,x + f_r,$$

Property:

$$\frac{e_i}{2b_r} = \frac{d_i}{4c_r} = \frac{c_i}{2a_r} = m\alpha_r\sigma^{-r-2s}. \to \eta(t) = \exp\left[-m\alpha_r\sigma^{-r-2s}x^2\right],$$

The Dyson map and metric

All invariants can be written as $(\gamma \neq 0)$

$$I_H = a_r p^2 + b_r p + (c_r + ic_i)\{x, p\} + (d_r + id_i)x^2 + (e_r + ie_i)x + f_r,$$

Property:

$$\frac{e_i}{2b_r} = \frac{d_i}{4c_r} = \frac{c_i}{2a_r} = m\alpha_r\sigma^{-r-2s}. \rightarrow \eta(t) = \exp\left[-m\alpha_r\sigma^{-r-2s}x^2\right],$$

Substitute $\eta(t)$ into TDDE

$$h = \frac{\sigma^{r+2s}}{2m}p^2 + \left(2m\alpha_i^2\sigma^{-r-2s} + \frac{1}{2}m\sigma^{-r-2s}\Omega^2\right)x^2 + \frac{1}{4}\partial_t \ln\left(\frac{\sigma^{r+2s}}{\alpha_r}\right)\{x, p\}.$$

.

Conclusions

- Point transformations can be used to construct non-Hermitian invariants for time-dependent non-Hermitian systems.

[6] A. Fring and R. Tenney, arXiv:2108.06793 [quant-ph]
[7] A. Fring and R.Tenney, Phys. Lett. A. **410** 127548 (2021)

Conclusions

- Point transformations can be used to construct non-Hermitian invariants for time-dependent non-Hermitian systems.
- Increase number of steps to obtain Dyson map, bypass ansatz for invariant.

[6] A. Fring and R. Tenney, arXiv:2108.06793 [quant-ph]

[7] A. Fring and R.Tenney, Phys. Lett. A. **410** 127548 (2021)

Conclusions

- Point transformations can be used to construct non-Hermitian invariants for time-dependent non-Hermitian systems.
- Increase number of steps to obtain Dyson map, bypass ansatz for invariant.
- Easier to determine Dyson maps $\rightarrow$ easier to determine multiple maps $\rightarrow$ construct an infinite series of Dyson maps[6].

[6] A. Fring and R. Tenney, arXiv:2108.06793 [quant-ph]

[7] A. Fring and R.Tenney, Phys. Lett. A. **410** 127548 (2021)

Conclusions

- Point transformations can be used to construct non-Hermitian invariants for time-dependent non-Hermitian systems.
- Increase number of steps to obtain Dyson map, bypass ansatz for invariant.
- Easier to determine Dyson maps $\rightarrow$ easier to determine multiple maps $\rightarrow$ construct an infinite series of Dyson maps[6].
- Reference Hamiltonian need not be exactly solvable to obtain invariant for target Hamiltonian. Reference Hamiltonian doesn't need to be Hermitian[7].

[6] A. Fring and R. Tenney, arXiv:2108.06793 [quant-ph]

[7] A. Fring and R.Tenney, Phys. Lett. A. **410** 127548 (2021)

Conclusions

- Point transformations can be used to construct non-Hermitian invariants for time-dependent non-Hermitian systems.
- Increase number of steps to obtain Dyson map, bypass ansatz for invariant.
- Easier to determine Dyson maps $\rightarrow$ easier to determine multiple maps $\rightarrow$ construct an infinite series of Dyson maps[6].
- Reference Hamiltonian need not be exactly solvable to obtain invariant for target Hamiltonian. Reference Hamiltonian doesn't need to be Hermitian[7].
- Apply technique to determine Dyson maps for more complicated systems.

Thank you for you attention.

[6] A. Fring and R. Tenney, arXiv:2108.06793 [quant-ph]

[7] A. Fring and R.Tenney, Phys. Lett. A. **410** 127548 (2021)