Time-dependent pseudo-squeezed coherent states

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Abstract

- We present the definition and properties of the time-dependent pseudo-squeezed coherent states.
- We introduce a pseudo-squeezed bosonic ladder operators obtained from the squeezed transformation of the pseudo- bosons annihilation and creation operators defined as a time-dependent non-Hermitian linear integral of motion.
- The pseudo-squeezed coherent states are obtained by applying the pseudo-displacement operator on the pseudo-squeezed vacuum where the exponent of pseudo-squeezed displacement operator is linear in pseudo-squeezed bosonic ladder operator.
- As illustration, we study the time-dependent non-Hermitian Swanson oscillator.

- **Introduction**

*Coherent and squeezed coherent states for Hermitian systems

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- **Time-dependent pseudo-squeezed coherent states**

*Time-dependent pseudo-bosons and pseudo-coherent states

*Pseudo-squeezed coherent states

- **Application: Time-dependent Swanson Hamiltonian**

*Pseudo-squeezed coherent states in position representation

The coherent states $|\alpha\rangle$ for the simple harmonic oscillator are (1) destruction operator eigenstates,

$$a|\alpha\rangle = \alpha|\alpha\rangle, \quad \alpha \in \mathbb{C} \tag{1}$$

(2) created from the ground state by a unitary operator (Glauber displacement operator) $D(\alpha)$

$$|\alpha\rangle = D(\alpha)|0\rangle = \exp\left[\left(\alpha a^{\dagger} - \alpha^{*} a\right)\right]|0\rangle, \tag{2}$$

and (3) They minimise the uncertainty relations

$$\Delta x \Delta p \geq \frac{1}{2}. \tag{3}$$

The coherent states $|\alpha\rangle$ are normalized but not orthogonal, and they are complete :

$$\frac{1}{\pi}\int |\alpha\rangle\langle\alpha| d^2\alpha = I, \tag{4}$$

In quantum physics, the squeeze operator for a single mode of the electromagnetic field is

$$S(\xi) = \exp\left[\frac{1}{2}(\xi a^{\dagger 2} - \xi^* a^2)\right] , \quad \xi \in \mathbb{C}. \tag{5}$$

Its action on the annihilation and creation operators produces

$$\begin{aligned} b &= S(\xi) a S^{-1}(\xi) = \cosh|\xi| \; a - \frac{\xi}{|\xi|} \sinh|\xi| \, a^\dagger \\ b^\dagger &= S(\xi) a^\dagger S^{-1}(\xi) = \cosh|\xi| \; a^\dagger - \frac{\xi^*}{|\xi|} \sinh|\xi| \, a. \end{aligned} \tag{6}$$

The squeeze operator produces the squeezed vacuum state $|\xi\rangle$ when acting on the vacuum $|0\rangle$,

$$|\xi\rangle = S(\xi)\,|0\rangle\,. \tag{7}$$

The squeeze operator can also act on coherent states and produce squeezed coherent states. The squeezing operator $S(\xi)$ does not commute with the displacement operator $D(\alpha)$: $S(\xi)D(\alpha) \neq D(\alpha)S(\xi)$. There is, however, a simple braiding relation,

$$D(\alpha)S(\xi) = S(\xi)D(\gamma), \quad \gamma(\alpha, \xi) = \alpha \cosh|\xi| - \alpha^* \frac{\xi}{|\xi|} \sinh|\xi| \,. \tag{8}$$

Therefore, the ordering of D and S is only a convention. Application of both operators above on the vacuum produces squeezed coherent states:

$$D(\alpha)S(\xi)\,|0\rangle \equiv |\alpha, \xi\rangle \,. \tag{9}$$

Note that for $\xi = 0$, it becomes an ordinary coherent state $|\alpha\rangle$. Taking into consideration that $0 = S(\xi)D(\gamma)a\,|0\rangle = S(\xi)D(\gamma)aD^{-1}(\gamma)S^{-1}(\xi)\,|\alpha, \xi\rangle = (b - \gamma)\,|\alpha, \xi\rangle$, we see that the squeezed coherent states are the eigenstates of the transformed annihilation operator,

$$b\,|\alpha, \xi\rangle = \gamma(\alpha, \xi)\,|\alpha, \xi\rangle \,, \tag{10}$$

which can be regarded as an alternative definition of these squeezed coherent states.

A method for constructing coherent states for an abritrary quantum dynamical system based on the employment of integral of motion $\mathcal{A}(t) = u(t)a + v(t)a^{\dagger}$ where $u(t)$, and $v(t)$ are complex functions satisfying the condition $|u(t)|^2 - |v(t)|^2 = 1$. The operator $\mathcal{A}(t)$ is an integral of motion if it obeys to the equation

$$\frac{\partial \mathcal{A}(t)}{\partial t} - i\left[\mathcal{A}(t), h(t)\right] = 0, \tag{11}$$

In the case where the Hamiltonian $h(t)$ is self-adjoint, the adjoint operator$\mathcal{A}^{\dagger}(t)$ is also an integral of motion,

$$\frac{\partial \mathcal{A}^{\dagger}(t)}{\partial t} - i\left[\mathcal{A}^{\dagger}(t), h(t)\right] = 0, \tag{12}$$

then one obtains two independent mutually Hermitian conjugate linear integrals of the motion satisfying the relation,

$$\left[\mathcal{A}(t), \mathcal{A}^{\dagger}(t)\right] = \mathbf{1}. \tag{13}$$

The operator integrals of motion $\mathcal{A}(t)$, $\mathcal{A}^{\dagger}(t)$ are unitarilly equivalent to the intial operators a, $a^{\dagger}$. Consequently, they possess the same properties. Thus it is natural to call the eigenstate of operator $\mathcal{A}(t)$ the "generalized" coherent state $|\alpha, t\rangle$: $\mathcal{A}(t)\, |\alpha, t\rangle = \alpha\, |\alpha, t\rangle$. Since $\mathcal{A}(t)$ is an invariant operator, the complex eigenvalue α does not depend on time t. Coherent states $|\alpha, t\rangle$ may be obtained also from the $\mathcal{A}(t)$-vaccum state $|0; t\rangle$: $\mathcal{A}(t)|0; t\rangle = 0$, by means of unitary operator $D(\alpha, t) = e^{(\alpha\mathcal{A}^{\dagger}(t) - \alpha^{*}\mathcal{A}(t))}$, i.e $|\alpha, t\rangle = D(\alpha, t)|0; t\rangle$.

Using the same strategy as the previous paragraghs, i.e., introducing the squeeze operator

$$S(\xi, t) = \exp\left[\frac{1}{2}(\xi\mathcal{A}^{2\dagger}(t) - \xi^{*}\mathcal{A}^{2}(t))\right] \ , \tag{14}$$

where the integrals of motion operators inside the exponential are the ladder operators. Its action on the annihilation $\mathcal{A}(t)$ and creation $\mathcal{A}^{\dagger}(t)$ operators produces

$$\mathcal{B}(t) = S(\xi,t)\mathcal{A}(t)S^{-1}(\xi,t) = \cosh|\xi|\ \mathcal{A}(t) - \frac{\xi}{|\xi|}\sinh|\xi|\,\mathcal{A}^{\dagger}(t)$$

$$\mathcal{B}^{\dagger}(t) = S(\xi,t)\mathcal{A}^{\dagger}(t)S^{-1}(\xi,t) = \cosh|\xi|\ \mathcal{A}^{\dagger}(t) - \frac{\xi^{*}}{|\xi|}\sinh|\xi|\,\mathcal{A}(t)\ . \tag{15}$$

The operators $\mathcal{B}(t)$ and $\mathcal{B}^{\dagger}(t)$ are also integrals of motions and can be considered as ladder operators. The squeeze operator, when acting upon the $\mathcal{A}(t)$-vaccum state vacuum $|0;t\rangle$, produces the squeezed vacuum state $|\xi,t\rangle$,

$$|\xi,t\rangle = S(\xi,t)\,|0,t\rangle\ . \tag{16}$$

The squeezing operator can also act on coherent states $|\alpha,t\rangle$ and produce squeezed coherent states. The squeezing operator $S(\xi,t)$ does not commute with the displacement operator $D(\alpha,t)$, there is, however, a simple braiding relation,

$$D(\alpha,t)S(\xi,t) = S(\xi)D(\gamma,t),\ \text{where}\ \gamma(\alpha,\xi) = \alpha\cosh|\xi| - \alpha^{*}\frac{\xi}{|\xi|}\sinh|\xi|\,. \tag{17}$$

Therefore, application of both operators above on the vacuum produces squeezed coherent states:

$$D(\alpha, t)S(\xi, t) |0, t\rangle \equiv |\alpha, \xi, t\rangle . \tag{18}$$

Note that for $\xi = 0$, it becomes an ordinary coherent state $|\alpha, t\rangle$. Studies of non-self-adjoint Hamiltonian are interesting from a physical point of view. In general, the representation space of such non-Hermitian systems can be equipped with a bi-orthogonal structure that provides complete sets of orthonormal states. It has been established that a non-Hermitian Hamiltonian can be connected to its Hermitian conjugate, $H^\dagger = \eta H \eta^{-1}$, through a linear, invertible, Hermitian and positive metric operator $\eta = \rho^\dagger \rho$, i.e. H is η -pseudo-Hermitian with respect to a positive-definite inner product defined by $\langle ., .\rangle_\eta = \langle . |\eta| .\rangle$. More recently a considerable attention is paid to an alternative formalism for description of non-Hermitian systems, based on the concept of the so-called pseudo-bosons.

Pseudo-bosons are a pseudo-Hermitian extension of usual bosons and are obtained from the modification of bosons commutation relations as follows

$$[A, \bar{A}] = [\bar{A}^{\dagger}, A^{\dagger}] = 1 \ , \tag{19}$$

where the operators A and $\bar{A}$ are related to their adjoint operator $\bar{A}^{\dagger}$ and $A^{\dagger}$ via the bounded Hermitian invertible operator or metric operator η as

$$\bar{A}^{\dagger} = \eta A \eta^{-1}, \ \ A^{\dagger} = \eta \bar{A} \eta^{-1}. \tag{20}$$

Thus the pseudo-coherent states for the pseudo-Hermitian boson systems are introduced and defined as eigenstates of the corresponding pseudo-boson annihilation operators A and $\bar{A}^{+}$

$$A \left| \psi_{\alpha} \right\rangle = \alpha \left| \psi_{\alpha} \right\rangle , \quad \bar{A}^{\dagger} \left| \phi_{\alpha} \right\rangle = \alpha \left| \phi_{\alpha} \right\rangle , \quad \left| \phi_{\alpha} \right\rangle = \eta \left| \psi_{\alpha} \right\rangle , \ \alpha \in \mathbb{C}. \tag{21}$$

and satisfy the resolution of the identity

$$\frac{1}{\pi} \int_C \left| \phi_{\alpha} \right\rangle \langle \psi_{\alpha} | \, d\alpha^* d\alpha = \frac{1}{\pi} \int_C | \psi_{\alpha} \rangle \left\langle \phi_{\alpha} \right| d\alpha^* d\alpha = \mathbf{I}. \tag{22}$$

These pseudo-coherent states $\left|\psi_{\alpha}\right\rangle$ and $\left|\phi_{\alpha}\right\rangle$ can be generated respectively from the vacuum states $\left|\psi_{0}\right\rangle$ and $\left|\phi_{0}\right\rangle$ by the action of displacement operators $D(\alpha)$ and $\overline{D}^{\dagger}(\alpha)$ respectively,

$$
\begin{aligned}
D(\alpha)\left|\psi_{0}\right\rangle &= \left|\psi_{\alpha}\right\rangle = \exp(\alpha\bar{A} - \alpha^{*}A)\left|\psi_{0}\right\rangle, \\
\overline{D}^{\dagger}(\alpha)\left|\phi_{0}\right\rangle &= \left|\phi_{\alpha}\right\rangle = \exp(\alpha A^{\dagger} - \alpha^{*}\bar{A}^{\dagger})\left|\phi_{0}\right\rangle
\end{aligned}
\tag{23}
$$

where $\overline{D}^{\dagger}(\alpha) = \eta D(\alpha)\eta^{-1}$ is the complementary pseudo-unitary displacement operator of $D(\alpha)$.

Here we address the problem of construction of **time-dependent** pseudo-Hermitian boson (pseudo-boson) creation and annihilation operators and pseudo-coherent states. To do this, we introduce quantum non-Hermitian integrals of motion, namely $A(t)$ and $\bar{A}(t)$ associated to the time-dependent non-Hermitian Hamiltonian $H(t)$, and $(\bar{A}^{\dagger}(t)\ ,\ A^{\dagger}(t))$ associated to its adjoint $H^{\dagger}(t)$, verifying the following equations:

$$\frac{\partial A(t)}{\partial t} - i\left[A(t), H(t)\right] = 0, \quad \frac{\partial \bar{A}(t)}{\partial t} - i\left[\bar{A}(t), H(t)\right] = 0, \qquad (24)$$

$$\frac{\partial \bar{A}^{\dagger}(t)}{\partial t} - i\left[\bar{A}^{\dagger}(t), H^{\dagger}(t)\right] = 0, \quad \frac{\partial A^{\dagger}(t)}{\partial t} - i\left[A^{\dagger}(t), H^{\dagger}(t)\right] = 0, \qquad (25)$$

where the Hamiltonian $H(t)$ governs the time-dependent Schrödinger equation

$$i\partial_t \left|\Psi^H(t)\right\rangle = H(t)\left|\Psi^H(t)\right\rangle, \tag{26}$$

and the solutions $\left|\Psi^H(t)\right\rangle$ are equal to the eigenstates $|\psi(t)\rangle$ of the invariant operator $A(t)$ multiplied by a time-dependent global phase factor

$$\left|\Psi^H(t)\right\rangle = e^{i\varphi(t)}\,|\psi(t)\rangle\,. \tag{27}$$

In order to construct pseudo-bosonic coherent states we consider that,

$$[A(t), \bar{A}(t)] = [\bar{A}^\dagger(t), A^\dagger(t)] = I. \tag{28}$$

The operators $A(t)$ ($\bar{A}(t)$) associated to $H(t)$ are related to the operators $\bar{A}^\dagger(t)$ ($A^\dagger(t)$) associated to $H^\dagger(t)$ via the time-dependent bounded Hermitian invertible operator $\eta(t)$ as

$$A(t) = \eta^{-1}(t)\bar{A}^\dagger(t)\eta(t)\ \ ,\ \ \bar{A}(t) = \eta^{-1}(t)A^\dagger(t)\eta(t). \tag{29}$$

The pseudo-bosonic coherent states are generated by the action on the vacuum states $\left\{\left|\psi_{_0}(t)\right\rangle, |\phi_0(t)\rangle\right\}$ of the pseudo-displacement operators $\{D^H(\alpha,t), D^{H^\dagger}(\alpha,t)\}$

$$\left|\psi_{_\alpha}(t)\right\rangle = D^H(\alpha,t)\,|\psi_0(t)\rangle = \exp(\alpha\bar{A}(t) - \alpha^* A(t))\,|\psi_0(t)\rangle\,, \tag{30}$$

$$\left|\phi_{_\alpha}(t)\right\rangle = D^{H^\dagger}(\alpha,t)\,|\phi_0(t)\rangle\,. \tag{31}$$

Note that $D^{H^\dagger}(\alpha,t)$ is the pseudo-adjoint of $D^H(\alpha,t)$, i.e.

$$D^{H^\dagger}(\alpha,t) = \eta(t)D^H(\alpha,t)\eta^{-1}(t) = \exp(\alpha A^\dagger(t) - \alpha^*\bar{A}^\dagger(t)).$$

The vacuum states are defined by

$$A(t)\left|\psi_{_0}(t)\right\rangle = 0, \quad \bar{A}^\dagger(t)\left|\phi_{_0}(t)\right\rangle = 0. \tag{32}$$

Consequently, the vacuum states $|\psi_0(t)\rangle$ and $|\phi_0(t)\rangle$ are related to each other as

$$\left|\phi_{_0}(t)\right\rangle = \eta\,(t)\left|\psi_{_0}(t)\right\rangle. \tag{33}$$

The same expressions for $\left\{ \left| \psi_{\alpha}(t) \right\rangle, \left| \phi_{\alpha}(t) \right\rangle = \eta(t) \left| \psi_{\alpha}(t) \right\rangle \right\}$ can be obtained by defining them as an eigenstates of the annihilation operators $\left\{ A(t), \bar{A}^{\dagger}(t) \right\}$ with a complex time-independent eigenvalue α, i.e.,

$$A(t) \left| \psi_{\alpha}(t) \right\rangle = \alpha \left| \psi_{\alpha}(t) \right\rangle, \quad \bar{A}^{\dagger}(t) \left| \phi_{\alpha}(t) \right\rangle = \alpha \left| \phi_{\alpha}(t) \right\rangle. \tag{34}$$

In particular, the choice of the normalization condition as $\left\langle \psi_{0}(t) \right| \eta(t) \left| \psi_{0}(t) \right\rangle = 1$ lead to

$$\left\langle \psi_{\alpha}(t) \right| \eta(t) \left| \psi_{\alpha}(t) \right\rangle = 1 \tag{35}$$

and, then the integral

$$\frac{1}{\pi} \int_{\mathcal{C}} \eta(t) \left| \psi_{\alpha}(t) \right\rangle \left\langle \psi_{\alpha}(t) \right| d\alpha^{*} d\alpha = \frac{1}{\pi} \int_{\mathcal{C}} \left| \phi_{\alpha}(t) \right\rangle \left\langle \phi_{\alpha}(t) \right| \eta^{-1}(t) d\alpha^{*} d\alpha = I. \tag{36}$$

Another important class of quantum states are the squeezed states which are generated by the action of the squeezing operator,

$$S^H(\xi, t) = \exp\frac{1}{2}(\xi\bar{A}^2(t) - \xi^* A^2(t)), \tag{37}$$

on the vacuum state $\left|\psi_0(t)\right\rangle$ of $A(t)$, also this is the squeezed vacuum state, which we denote as $|\xi, t\rangle$ i.e;

$$|\xi, t\rangle = S^H(\xi, t)\left|\psi_0(t)\right\rangle, \tag{38}$$

where ξ is the complex squeeze time-independent parameter.
The squeeze operators are obtained as

$$B(t)=S^{H}(\xi,t)A(t)S^{-1H}(\xi,t)=\cosh|\xi|\;A(t)-\frac{\xi}{|\xi|}\sinh|\xi|\,\bar{A}(t),$$
$$\bar{B}(t)=S^{H}(\xi,t)\bar{A}(t)S^{-1H}(\xi,t)=\cosh|\xi|\;\bar{A}(t)-\frac{\xi^{*}}{|\xi|}\sinh|\xi|\,A(t). \tag{39}$$

Note that $B(t)$ and $\bar{B}(t)$ are a linear combination of $A(t)$ and $\bar{A}(t)$, consequently they can be considered as new invariant operators verifying the following equations

$$\frac{\partial B(t)}{\partial t}-i\left[B(t),H(t)\right]=0,\quad \frac{\partial \bar{B}(t)}{\partial t}-i\left[\bar{B}(t),H(t)\right]=0. \tag{40}$$

The ladder operators $B(t)$ $(\bar{B}(t))$ associated to $H(t)$ are related to the operators $\bar{B}^{\dagger}(t)$ $(B^{\dagger}(t))$ associated to $H^{\dagger}(t)$ via the time-dependent bounded Hermitian invertible operator $\eta(t)$ as

$$B(t)=\;\eta^{-1}(t)\bar{B}^{\dagger}(t)\eta(t)\;\;,\;\;\bar{B}(t)=\;\eta^{-1}(t)B^{\dagger}(t)\eta(t). \tag{41}$$

In order to construct the time-dependent pseudo-bosonic squeezed coherent states, we consider the invariant operators $B(t)$ and $\bar{B}(t)$ ($\bar{B}^{\dagger}(t)$ and $B^{\dagger}(t)$) as time-dependent pseudo-bosonic squeeze ladder operators associated to $H(t)$ ($H^{\dagger}(t)$), respectively, that verify the commutation relations

$$[B(t), \bar{B}(t)] = \left[\bar{B}^{\dagger}(t), B^{\dagger}(t)\right] = I. \tag{42}$$

We may define squeezed states in an alternative way where we start from the squeezed vacuum (38)

$$B(t)\,|\xi, t\rangle = S^{H}(\xi, t)A(t)\left|\psi_{_0}(t)\right\rangle = 0. \tag{43}$$

A more general pseudo-bosonic squeezed coherent states $\left|\psi_{_{\alpha,\xi}}(t)\right\rangle$ may be obtained by applying the the pseudo-squeezed displacement operator $T(\gamma, t)$ to the squeezed vacuum state $|\xi, t\rangle$:

$$\left|\psi_{\alpha,\xi}(t)\right\rangle = T(\gamma, t)\,|\xi, t\rangle = \exp(\gamma\bar{B}(t) - \gamma^{*}B(t))\,|\xi, t\rangle\,. \tag{44}$$

Obviously for $\xi = 0$ we obtain just a pseudo-coherent state.

The properties of the pseudo-squeezed coherent states $\left|\psi_{\alpha,\xi}(t)\right\rangle$ may be proved to parallel those of the pseudo-coherent states $\left|\psi_{\alpha}(t)\right\rangle$. Since our pseudo-squeezed coherent state are closely related to the ones of the pseudo-coherent states $\left|\psi_{\alpha}(t)\right\rangle$, other constructions of squeezed coherent states can be considered using the ladder operators $B(t)$ (39), it follows that

$$B(t)\left|\psi_{\alpha,\xi}(t)\right\rangle = \gamma(\alpha,\xi)\left|\psi_{\alpha,\xi}(t)\right\rangle. \tag{45}$$

$$\gamma(\alpha,\xi) = \cosh|\xi|\,\alpha - \frac{\xi}{|\xi|}\sinh|\xi|\,\alpha^{*}. \tag{46}$$

The equality in Eq. (45) comes from the relations

$$T(\gamma,t)B(t)T^{-1}(\gamma,t) = B(t)-\gamma, \quad T(\gamma,t)B^{\dagger}(t))T^{-1}(\gamma,t) = B^{\dagger}(t)-\gamma^{*}. \tag{47}$$

Let us next turn our attention to some questions which arise from the above equation (44). Use of the properties of the squeeze operator given in equation (39) leads to

$$T(\gamma,t) = S^{H}(\xi,t)D^{H}(\gamma,t)S^{-1H}(\xi,t) = D^{H}(\alpha,t). \tag{48}$$

A pseudo-squeezed coherent state $\left|\psi_{\alpha,\xi}(t)\right\rangle$ is obtained by first acting with the pseudo-squeezed displacement operator $T(\gamma,t)$ on the pseudo-squeezed vacuum states $|\xi,t\rangle$ or with the displacement operator $D^H(\alpha,t)$ on the pseudo-squeezed vacuum states $|\xi,t\rangle$. On the other hand, when acting with the pseudo- squeezed displacement operator $T(\gamma,t)$ on the pseudo vacuum $\left|\psi_{_0}(t)\right\rangle$, we obtain the pseudo-bosonic coherent states $|\psi_\alpha(t)\rangle$.

Knowing that, the pseudo-vacuum states $\{|\psi_0(t)\rangle, |\phi_0(t)\rangle\}$ of $\{A(t), \bar{A}^\dagger(t)\}$ respectively are related to each other as $\left|\phi_{_0}(t)\right\rangle = \eta(t)\left|\psi_{_0}(t)\right\rangle$, consequently the pseudo-squeezed vacuum states $|\xi,t\rangle$ and $\widetilde{|\xi,t\rangle}$ of $B(t)$ and $\bar{B}^\dagger(t)$ are linked to each other as

$$\widetilde{|\xi,t\rangle} = \eta(t)\,|\xi,t\rangle = \left[S^H(\xi,t)\right]^{-1\dagger}\left|\phi_{_0}(t)\right\rangle. \tag{49}$$

Pseudo-bosonic squeezed coherent state $\left|\phi_{_{\alpha,\xi}}(t)\right\rangle$, associated to $H^\dagger(t)$, can also be obtained from the action of the displacement operator $T^{H^\dagger}(\gamma,t) = \left[\eta(t)T(\gamma,t)\eta^{-1}(t)\right] = \left[T(\gamma,t)\right]^{-1\dagger}$ on the pseudo-squeezed vacuum state $\widetilde{|\xi,t\rangle}$ of $\bar{B}^\dagger(t)$ as

$$\left|\phi_{\alpha,\xi}(t)\right\rangle = \left[T(\gamma,t)\right]^{-1\dagger}\widetilde{|\xi,t\rangle} = \exp(\gamma B^{\dagger}(t) - \gamma^{*}\bar{B}^{\dagger}(t))\widetilde{|\xi,t\rangle}. \quad (50)$$

The pseudo-squeezed coherent states $\left|\phi_{\alpha,\xi}(t)\right\rangle$ are eigenstates of the operator $\bar{B}^{\dagger}(t)$ with the complex time-independent eigenvalue γ

$$\bar{B}^{\dagger}(t)\left|\phi_{\alpha,\xi}(t)\right\rangle = \gamma\left|\phi_{\alpha,\xi}(t)\right\rangle. \quad (51)$$

Thus, the normalization condition $\left\langle\psi_{0}(t)\right|\eta(t)\left|\psi_{0}(t)\right\rangle = I$ lead to

$$\left\langle\psi_{\alpha,\xi}(t)\right|\eta(t)\left|\psi_{\alpha,\xi}(t)\right\rangle = \left\langle\phi_{\alpha,\xi}(t)\right.\left|\psi_{\alpha,\xi}(t)\right\rangle = I, \quad (52)$$

which show that the pseudo-bosonic squeezed coherent states form an overcomplete set in that the identity can be resolved as

$$\frac{1}{\pi}\int_{\mathcal{C}}\eta(t)\left|\psi_{\alpha,\xi}(t)\right\rangle\left\langle\psi_{\alpha,\xi}(t)\right| d\gamma^{*}d\gamma = I. \quad (53)$$

Application: Time-dependent Swanson Hamiltonian

Let us consider the non-Hermitian Swanson oscillator described by the Hamiltonian

$$H(t) = \omega(t)\left(a^{+}a + \frac{1}{2}\right) + \beta(t)a^{2} + \lambda(t)a^{+2}, \qquad (54)$$

where a and $a^{\dagger}$ are bosonic annihilation and creation operators of a light field mode verifying $\left[a, a^{\dagger}\right] = 1$. The coefficients $\{\omega(t), \beta(t), \lambda(t)\}$ are time-dependent complex parameters.
As a starting point, let the linear non-Hermitian pseudo-bosonic invariant operator be in the form

$$A(t) = \left[\delta_1(t)a + \delta_2(t)a^{\dagger} + \delta_3(t)\right], \qquad \delta_1(t) \neq \delta_2(t) \in R, \qquad (55)$$

the invariance condition (24) leads to the following equations:

$$\begin{cases} \dot{\delta}_1(t) = i\left(\omega(t)\delta_1(t) - 2\beta(t)\delta_2(t)\right) \\ \dot{\delta}_2(t) = i\left(2\lambda(t)\delta_1(t) - \omega(t)\delta_2(t)\right) \end{cases}, \quad \dot{\delta}_3(t) = 0. \qquad (56)$$

Without loss of generality, we set $\delta_3(t) = 0$. Then separating the Eqs.(56) into real part and imaginary part, we find the following coupled equations for the parameters $\delta_1(t)$ and $\delta_2(t)$:

$$\delta_1 |\omega| \cos\varphi_\omega - 2\delta_2 |\beta| \cos\varphi_\beta = 0, \tag{57}$$

$$2\delta_1 |\lambda| \cos\varphi_\gamma - \delta_2 |\omega| \cos\varphi_\omega = 0, \tag{58}$$

$$\delta_1(t) = \exp\left[\int_0^t |\omega(t')| \cos\varphi_\omega(t') \left(\tan\varphi_\beta(t') - \tan\varphi_\omega(t')\right) dt'\right], \tag{59}$$

$$\delta_2(t) = \exp\left[\int_0^t |\omega| \cos\varphi_\omega \left(\tan\varphi_\omega - \tan\varphi_\gamma\right) dt'\right]. \tag{60}$$

In order to solve the time-dependent quasi-Hermiticity relation (29) we make an ansatz for the time-dependent metric $\eta(t)$

$$\eta(t) = \exp\left[\frac{1}{2}\vartheta_+(t)\, a^{\dagger 2}\right] \exp\left[\frac{1}{2}\ln\vartheta_0(t)\left(a^\dagger a + \frac{1}{2}\right)\right] \exp\left[\frac{1}{2}\vartheta_-(t)\, a^2\right], \tag{61}$$

where

$$\vartheta_+(t) = \frac{2(2\mu^*)\sinh\theta}{\theta\cosh\theta - 2\epsilon\sinh\theta} = -\zeta(t)e^{-i\varphi(t)},$$
$$\vartheta_0(t) = \left(\cosh\theta - \frac{2\epsilon}{\theta}\sinh\theta\right)^{-2} = \zeta^2(t) - \chi(t), \quad \theta = 2\sqrt{\epsilon^2 - 4|\mu|^2}, \tag{62}$$

$$\vartheta_-(t) = \frac{2(2\mu)\sinh\theta}{\theta\cosh\theta - 2\epsilon\sinh\theta} = -\zeta(t)e^{+i\varphi(t)},$$
$$\chi(t) = -\frac{\cosh\theta + \frac{2\epsilon}{\theta}\sinh\theta}{\cosh\theta - \frac{2\epsilon}{\theta}\sinh\theta},$$

we deduce

$$\begin{cases} \eta(t)\, a\eta^{-1}(t) = \frac{1}{\sqrt{\vartheta_0}}(a - \vartheta_+ a^\dagger) \\ \eta(t)\, a^\dagger\eta^{-1}(t) = \frac{1}{\sqrt{\vartheta_0}}\left(a\vartheta_- - \chi a^\dagger\right) \end{cases}. \tag{63}$$

Therefore, the pseudo-bosonic operator $\bar{A}(t)$ determined from the quasi-Hermiticity realtion (29) has the form

$$\bar{A}(t) = \frac{1}{\sqrt{\vartheta_0}} \left[\delta_1 + \vartheta_+ \delta_2\right] a^\dagger - \frac{1}{\sqrt{\vartheta_0}} \left[\vartheta_- \delta_1 + \chi \delta_2\right] a. \quad (64)$$

Knowing that the operators $A(t)$ and $\bar{A}\,(t)$ are, by the assumption (28), annihilation and creation operators, this leads to

$$\left[\delta_1^2\,(t) + \chi \delta_2^2\,(t) + (\vartheta_+ + \vartheta_-)\,\delta_1\,(t)\,\delta_2\,(t)\right] = \sqrt{\vartheta_0}. \quad (65)$$

Substituting representation (64) into (24) and using Eqs. (57) and (58) , we obtain the following equation for the parameters $\zeta(t), \chi(t)$ and $\varphi(t)$

$$\begin{aligned}&(\chi\delta_2^2 - \delta_1^2)\frac{\dot{\vartheta}_0}{2\vartheta_0} - \delta_2^2\dot{\chi} \\ &= 2(\delta_1^2 + \chi\delta_2^2)\,|\omega| \sin\varphi_\omega - 4\zeta\cos\varphi\left[\delta_1^2\,|\gamma| \sin\varphi_\gamma + \delta_2^2\,|\beta| \sin\varphi_\beta\right]. \quad (66)\end{aligned}$$

Now, by using Eq. (39) we are in position to show that $S^H(\xi, t)$ induces a transformation of the creation and annihilation squeeze operators as

$$
\begin{aligned}
B(t) &= \left(\delta_1(t) \cosh|\xi| + \frac{\xi}{|\xi|} \frac{1}{\sqrt{\vartheta_0}} \sinh|\xi| \; [\vartheta_- \delta_1(t) + \chi \delta_2(t)] \right) a \\
&\quad + \left(\delta_2(t) \cosh|\xi| - \frac{\xi}{|\xi|} \frac{1}{\sqrt{\vartheta_0}} \sinh|\xi| \, [\delta_1(t) + \vartheta_+ \delta_2(t)] \right) a^\dagger, \\
\bar{B}(t) &= \left(\frac{1}{\sqrt{\vartheta_0}} \cosh|\xi| \, [\delta_1(t) + \vartheta_+ \delta_2(t)] - \frac{\xi^*}{|\xi|} \sinh|\xi| \, \delta_2(t) \right) a^\dagger \\
&\quad - \left(\frac{1}{\sqrt{\vartheta_0}} \cosh|\xi| \; [\vartheta_- \delta_1(t) + \chi \delta_2(t)] - \frac{\xi^*}{|\xi|} \sinh|\xi| \, \delta_1(t) \right) a. \qquad (67)
\end{aligned}
$$

Let us express the Eqs. (55), (64) and (67) in position and momentum operators representation for the case where $a = \frac{1}{\sqrt{2}}(x + ip)$, and $a^\dagger = \frac{1}{\sqrt{2}}(x - ip)$:

$$A(t) = [fx + igp] \tag{68}$$
$$\bar{A}(t) = \left[\tilde{f}x - i\tilde{g}p\right] \tag{69}$$

where

$$\begin{cases} f(t) = \frac{1}{\sqrt{2}}(\delta_1(t) + \delta_2(t)) \\ g(t) = \frac{1}{\sqrt{2}}(\delta_1(t) - \delta_2(t)) \end{cases}, \tag{70}$$

$$\begin{cases} \tilde{f}(t) = \frac{1}{\sqrt{2\vartheta_0}}\left[(1 - \vartheta_-)\delta_1 + (\vartheta_+ - \chi)\delta_2\right] \\ \tilde{g}(t) = \frac{1}{\sqrt{2\vartheta_0}}\left[(1 + \vartheta_-)\delta_1 + (\vartheta_+ + \chi)\delta_2\right] \end{cases}, \tag{71}$$

the condition (28) gives $f(t)\tilde{g}(t) + g(t)\tilde{f}(t) = 1$.

In addition, the operators $B(t)$and $\bar{B}(t)$ can also be written as

$$\begin{aligned} B(t) &= \left[f_\xi x + i g_\xi p\right] \\ \bar{B}(t) &= \left[\tilde{f}_\xi x - i \tilde{g}_\xi p\right] \end{aligned} \tag{72}$$

where

$$f_\xi = \cosh|\xi| \, f - \frac{\xi}{|\xi|} \sinh|\xi| \, \tilde{f} \tag{73}$$

$$g_\xi = \cosh|\xi| \, g + \frac{\xi}{|\xi|} \sinh|\xi| \, \tilde{g} \tag{74}$$

and

$$\begin{aligned} \tilde{f}_\xi &= \cosh|\xi| \, \tilde{f} - \frac{\xi^*}{|\xi|} \sinh|\xi| \, f \\ \tilde{g}_\xi &= \cosh|\xi| \, \tilde{g} + \frac{\xi^*}{|\xi|} \sinh|\xi| \, g, \end{aligned} \tag{75}$$

the condition (42) imply that $f_\xi \tilde{g}_\xi + \tilde{f}_\xi g_\xi = 1$.

In order to construct the pseudo-squeezed coherent states in the position representation, the pseudo-squeezed vacuum state in the x-representation is required. This lead us to solve the eigenvalue equations $B(t)\,|\xi, t\rangle = 0$ and $\bar{B}^{\dagger}(t)\widetilde{|\xi, t\rangle} = 0$ in the x-representation

$$B(t)\,\langle x\,|\xi, t\rangle = \left(g_{\xi}\,(t)\,\frac{\partial}{\partial x} + f_{\xi}\,(t)\,x\right)\xi(x, t) = 0 \tag{76}$$

$$\bar{B}^{\dagger}(t)\,\langle x\,\widetilde{|\xi, t\rangle} = \left(\tilde{g}_{\xi}^{*}(t)\,\frac{\partial}{\partial x} + \tilde{f}_{\xi}^{*}(t)x\right)\widetilde{\xi(x, t)} = 0, \tag{77}$$

thus, the solutions of the above equations are

$$\xi(x, t) = \left(\frac{1}{2\pi\tilde{g}_{\xi}g_{\xi}}\right)^{\frac{1}{4}}\exp\left[-\frac{f_{\xi}}{2g_{\xi}}x^{2}\right] \tag{78}$$

and

$$\widetilde{\xi(x, t)} = \eta(t)\xi(x, t) = \left(\frac{1}{2\pi\tilde{g}_{\xi}^{*}g_{\xi}^{*}}\right)^{\frac{1}{4}}\exp\left[-\frac{\tilde{f}_{\xi}^{\,*}}{2\tilde{g}_{\xi}^{*}}x^{2}\right]. \tag{79}$$

where,

$$\langle \xi, t | \, \eta(t) \, | \xi, t \rangle = \int \widetilde{\xi^*(x,t)} \xi(x,t) dx = 1. \tag{80}$$

As pointed above, the pseudo-squeezed coherent state is obtained by the action of the displacement operator $T(\gamma, t)$ on the pseudo-squeezed vacuum (44). So by expessing $T(\gamma, t)$ in terms of

$$\begin{cases} x = g_{\xi} \bar{B} + \tilde{g}_{\xi} B \\ ip = \tilde{f}_{\xi} B - f_{\xi} \bar{B} \end{cases}, \tag{81}$$

we find

$$\begin{aligned} T(\gamma, t) &= \exp\left(-\frac{i}{2} \langle p \rangle_{\eta} \langle x \rangle_{\eta}\right) \exp\left(i \langle p \rangle_{\eta} x\right) \\ &\times \exp\left(-i \left[\frac{\langle x \rangle_{\eta} - \langle x \rangle_{\eta}^{*}}{2}\right] p\right) \\ &\times \exp\left(-i \left[\frac{\langle x \rangle_{\eta} + \langle x \rangle_{\eta}^{*}}{2}\right] p\right), \end{aligned} \tag{82}$$

where

$$\langle x\rangle_\eta = \left\langle \psi_{\alpha,\xi}\right| \eta x \left|\psi_{\alpha,\xi}\right\rangle = \gamma\tilde{g}_\xi + \gamma^* g_\xi \tag{83}$$

$$i\langle p\rangle_\eta = i\left\langle \psi_{\alpha,\xi}\right| \eta p \left|\psi_{\alpha,\xi}\right\rangle = \gamma\tilde{f}_\xi - \gamma^* f_\xi, \tag{84}$$

are complex numbers and not physically observables. By the same way, we deduce

$$\begin{aligned}\left[T^{-1}(\gamma,t)\right]^\dagger &= \exp\left(-\frac{i}{2}\langle p\rangle_\eta^*\langle x\rangle_\eta^*\right)\exp\left(i\langle p\rangle_\eta^* x\right)\\ &\quad\times\exp\left(-i\left[\frac{\langle x\rangle_\eta^*-\langle x\rangle_\eta}{2}\right]p\right)\\ &\quad\times\exp\left(-i\left[\frac{\langle x\rangle_\eta+\langle x\rangle_\eta^*}{2}\right]p\right).\end{aligned} \tag{85}$$

When the operators defined in (82) and (85) act on the pseudo-squeezed vacuums given in (78) and (79), we obtain the pseudo-squeezed coherent states

When the operators defined in (82) and (85) act on the pseudo-squeezed vacuums given in (78) and (79), we obtain the pseudo-squeezed coherent states

$$\begin{aligned}
\psi_{\alpha,\xi}(x,t) &= \left(\frac{1}{2\pi\tilde{g}_{\xi}g_{\xi}}\right)^{\frac{1}{4}} \exp\left[-\frac{i}{2}\langle p\rangle_{\eta}\langle x\rangle_{\eta}\right]\exp\left[i\langle p\rangle_{\eta}x\right] \\
&\times \exp\left(-i\left[\frac{\langle x\rangle_{\eta}-\langle x\rangle_{\eta}^{*}}{2}\right]p\right) \\
&\times \exp\left[-\frac{f_{\xi}}{2g_{\xi}}\left(x-\left[\frac{\langle x\rangle_{\eta}+\langle x\rangle_{\eta}^{*}}{2}\right]\right)^{2}\right]
\end{aligned} \tag{86}$$

and

$$
\begin{aligned}
\phi_{\alpha,\xi}(x,t) \quad = \quad & \left(\frac{1}{2\pi\tilde{g}_{\xi}^{*}g_{\xi}^{*}}\right)^{\frac{1}{4}} \exp\left[-\frac{i}{2}\left\langle p\right\rangle_{\eta}^{*}\left\langle x\right\rangle_{\eta}^{*}\right] \exp\left[i\left\langle p\right\rangle_{\eta}^{*} x\right] \\
& \times \exp\left(-i\left[\frac{\left\langle x\right\rangle_{\eta}^{*}-\left\langle x\right\rangle_{\eta}}{2}\right] p\right) \\
& \times \exp\left[-\frac{\tilde{f}_{\xi}^{*}}{2\tilde{g}_{\xi}^{*}}\left(x-\left[\frac{\left\langle x\right\rangle_{\eta}+\left\langle x\right\rangle_{\eta}^{*}}{2}\right]\right)^{2}\right]. \qquad (87)
\end{aligned}
$$

Therefore, the density $\left|\rho\left|\psi_{\alpha,\xi}(t)\right\rangle\right|^2 = \left\langle\phi_{\alpha,\xi}(t)\left|\psi_{\alpha,\xi}(t)\right\rangle\right.$ can be expressed as

$$\left|\rho(t)\psi_{\alpha,\xi}(x,t)\right|^2 = \left(\frac{1}{2\pi\tilde{g}_\xi g_\xi}\right)^{\frac{1}{2}} \exp\left[-\frac{1}{2\tilde{g}_\xi g_\xi}\left(x - \frac{\langle x\rangle_\eta + \langle x\rangle_\eta^*}{2}\right)^2\right] \tag{88}$$

and represents a Gaussian wave packet centered at $\left(x - \left[\frac{\langle x\rangle_\eta + \langle x\rangle_\eta^*}{2}\right]\right)$.

We see from this equation that the width of this Gaussian wave packet varies with time and is identical to $\sigma = \tilde{g}_\xi g_\xi$. It is also readily verified that the time-dependent pseudo-probability density is conserved:

$$\int dx \left(\psi_{\alpha,\xi}^*(x,t)\eta(t)\right)\psi_{\alpha,\xi}(x,t) = \int \phi_{\alpha,\xi}^*(x,t)\psi_{\alpha,\xi}(x,t)dx = I. \tag{89}$$

Thank you