Conformal symmetry plus PT-symmetry for perfect invisibility and related exotics

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Based on collaboration with

• Juan Mateos Guilarte (Salamanca U.):
JHEP **12** (2017) 061 [arXiv: 1602.02179]
JHEP **01** (2019) 194 [arXiv: 1710.00356]

+ related works with
Francisco Correa, Vit Jakubsky, Luismi Nieto, Mariano del Olmo, Adrian Arancibia, Olaf Lechtenfeld

and
• Luis Inzunza (USACH), Andreas Wipf (Jena U.):
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• Peculiar properties of various classical and quantum systems can be related to/derived from those of a free particle

• Free particle $\rightarrow$ Darboux transformations $\rightarrow$ reflectionless quantum systems

• Darboux covariance of Lax pair representation of the KdV equation

•$\Longrightarrow$ Potentials of reflectionless quantum systems can be promoted to multi-soliton solutions of the classical KdV equation

(• Inverse scattering theory: KdV evolution = isospectral deformation of the corresponding multi-soliton Schrödinger potential)

$\star$ $\star$ $\star$ potential of any reflectionless system is a snapshot of a multi-soliton solution to the KdV equation

- By periodization of reflectionless systems (at least of some of them), finite-gap quantum systems can be obtained; their potentials are solutions of stationary equations of the KdV hierarchy

- Darboux covariance of Lax representation also can be applied to finite-gap quantum systems to promote their potentials to cnoidal type solutions of the KdV equation

- Darboux transformations also can be applied to finite-gap systems to produce new solutions to the KdV and mKdV equations. Such systems represent soliton defects propagating in the periodic finite-gap background

- All these quantum systems are characterized by a nontrivial Lax-Novikov integral being a higher *odd order* differential operator

⋆ • ⋆ $\Rightarrow$ With all these quantum systems exotic nonlinear supersymmetry can be associated via Darboux transformations

⋆ • ⋆ In the simplest case, instead of $\mathcal{N}=2$ conventional supersymmetry described by Lie superalgebraic structure, there emerges exotic $\mathcal{N}=4$ nonlinear Poncaré supersymmetry, which involves Lax-Novikov integrals of stationary equations of the KdV hierarchy

⋆ ⋆ ⋆ Any two reflectionless (any two isospectral, or almost isospectral finite-gap) systems can be related by two different Darboux-Crum transformations

•$\Rightarrow$ Extended system is described by four supercharges instead of conventional two supercharges, and their anitcommutators generate not only (polynomial in) Hamiltonian of the extended system, but also Lax-Novikov integrals of their subsystems

• Lax-Novikov integrals = additional bosonic generators are higher odd order differential operators

• They separate left-right moving deformed plane waves of reflectionless systems in the continuous part of their spectra, or left- and right- moving Bloch states of finite-gap systems, and detect all the bound, or edge states in them

• Lax-Novikov integral of any reflectionless system is a Darboux dressed form of the momentum operator of the free particle:

$$\hat{\mathcal{P}} = \hat{\mathcal{D}}(\hat{p}\hat{\mathcal{D}}^{\dagger}) = (\hat{\mathcal{D}}\hat{p})\hat{\mathcal{D}}^{\dagger}$$

• On the other hand, Calogero-Moser systems govern the dynamics of moving poles of *rational* solutions of the KdV equation. These systems can be obtained via appropriate limit procedure from multi-soliton solutions, and by employing Galilean symmetry of the KdV equation

• They also can be obtained from the free particle via *singular* Darboux-Crum transformations

• The simplest case of two-particle Calogero systems corresponds to conformal mechanics: $\hat{H}_n = -\frac{d^2}{dx^2} + \frac{n(n+1)}{x^2}$, $n = 1, 2, \ldots$

• Schrödinger symmetry of a free particle reduces there to conformal $sl(2, \mathbb{R})$ symmetry

⋆ ⋆ ⋆ Lax-Novikov integral is lost, exotic SUSY is lost ⋆ ⋆ ⋆

• Schrödinger symmetry of two-particle Calogero system can be recuperated via $\mathcal{PT}$-regularization $x \to x + i\alpha$, $\alpha \in \mathbb{R}$, $\alpha \neq 0$

•⟹ $\mathcal{PT}$-regularization of Darboux transformations allows us to obtain perfectly invisible systems in which transmission coefficient, but not only its absolute value, equals one: $t(k) = 1$

•⟹ Another peculiarity: each of these systems contains a unique bound state of zero energy $E = 0$ at the very edge of the continuous doubly degenerate part of the spectrum

• Lax-Novikov integrals in the perfectly invisible $\mathcal{PT}$-regularized zero-gap quantum conformal and superconformal quantum mechanics systems affect on their (super)-conformal symmetries:

• Lax-Novikov integral modifies and extends conformal symmetry into a nonlinearly extended generalized Schrödinger algebra

• Exotic supersymmetric structure can be recuperated via $\mathcal{PT}$-regularization of Darboux transformations

Example

Seed state for the Darboux transformation :

$$\psi^{(1)}_{\alpha,\gamma} = \gamma\xi^{-1} + \xi^2\,, \qquad \xi = x + i\alpha\,, \qquad \alpha \in \mathbb{R}\,, \quad \alpha \neq 0\,,$$

$\gamma = 12t + i\nu\alpha^3$, $\nu \in (1,\infty)$; $\psi^{(1)}_{\alpha,\gamma}$ is a linear combination of the bound state ξ^{-1} of $H^{\alpha}_1 = -\frac{d^2}{dx^2} + \frac{2}{\xi^2}$ of zero eigenvalue and of its non-physical partner ξ^2.

$$\mathcal{W}^{(1)}_{\alpha,\gamma} = -\frac{d}{dx}\left(\ln\psi^{(1)}_{\alpha,\gamma}\right) = \frac{1}{\xi} - \frac{3\xi^2}{\xi^3+\gamma}\,,$$

$V_{\pm} = (\mathcal{W}^{(1)}_{\alpha,\gamma})^2 \pm (\mathcal{W}^{(1)}_{\alpha,\gamma})'$, $V_{-} = 2\xi^{-2}$ and

$$V_{+} = -2\left(\ln W(\xi, -\gamma + \xi^3)\right)'' = \frac{6}{\xi^2} - 6\gamma\frac{4\xi^3+\gamma}{\xi^2(\xi^3+\gamma)^2}\,.$$

$V_+(x;\alpha,\gamma)$ satisfies complexified KdV equation
$u_t - 6uu_x + u_{xxx} = 0$.
Its real and imaginary parts, $u(x,t) = v(x,t) + iw(x,t)$, satisfy the system of coupled equations
$v_t - 3(v^2 - w^2)_x + v_{xxx} = 0\,, \qquad w_t - 6(vw)_x + w_{xxx} = 0$

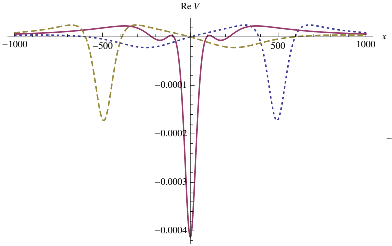


Figure: Evolution of real (on the left) and imaginary (on the right) parts of potential $u(x,t) = V_+(x;\alpha,\gamma(t))$ as a complex $\mathcal{PT}$-symmetric solution of the KdV equation at $\alpha = 100$, $\nu = 5$; dashed lines: $t = -10^7$, continuous lines: $t = 0$, dotted lines: $t = 10^7$.

$\psi^{(2)}_{\alpha,\gamma} = \gamma\xi^{-2} + \xi^3$ = zero energy eigenstate of $H_2^{\alpha} = -\frac{d^2}{dx^2} + \frac{6}{\xi^2}$ as a seed state for Darboux transformation

$$V_+^{(2)}(x;\alpha,\gamma) = -2\left(\ln W(\xi, \xi^3, \tfrac{8}{3}\gamma + \xi^5)\right)'' = \frac{12}{\xi^2} - 10\gamma\frac{6\xi^5+\gamma}{\xi^2(\xi^5+\gamma)^2}$$

With $\gamma = -720t + i\nu\alpha^5$, $\nu \in (24, \infty)$, it satisfies higher order equation of the KdV hierarchy

$$u_t + 30u^2u_x - 20u_xu_{xx} - 10uu_{xxx} + u_{xxxxx} = 0$$

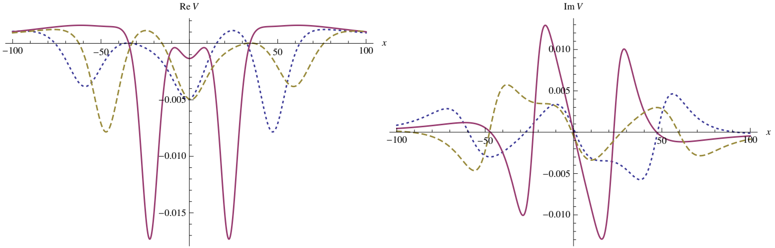


Figure: Evolution of real (on the left) and imaginary (on the right) parts of potential $V_+^{(2)}(x;\alpha,\gamma(t))$ at $\alpha = 20$, $\nu = 25$; dashed lines: $t = -10^6$, continuous lines: $t = 0$, dotted lines: $t = 10^6$

Simplest extended system :

$$\mathcal{H} = \begin{pmatrix} H_1^\alpha & 0 \\ 0 & H_0 \end{pmatrix}$$

Supercharges $([\mathcal{H}, Q_a] = 0\,,\ [\mathcal{H}, S_a] = 0)$:

$$Q_1 = \begin{pmatrix} 0 & D_1 \\ D_1^\# & 0 \end{pmatrix}, \qquad S_1 = \begin{pmatrix} 0 & -iD_1\mathcal{P}_0 \\ i\mathcal{P}_0 D_1^\# & 0 \end{pmatrix},$$

$Q_2 = i\sigma_3 Q_1$, $S_2 = i\sigma_3 S_1$, where $D_1 = \xi\frac{d}{dx}\xi^{-1} = \frac{d}{dx} - \xi^{-1}$, $D_1^\# = -\xi^{-1}\frac{d}{dx}\xi = -\frac{d}{dx} - \xi^{-1}$, $\mathcal{P}_0 = -i\frac{d}{dx}$ is the momentum operator of the free particle $H_0 = -\frac{d^2}{dx^2}$.

$$\{Q_a, Q_b\} = 2\delta_{ab}\mathcal{H}\,, \quad \{S_a, S_b\} = 2\delta_{ab}\mathcal{H}^2\,, \quad \{Q_a, S_b\} = 2\epsilon_{ab}\mathcal{L}_1\,,$$

$$\mathcal{L}_1 = \begin{pmatrix} \mathcal{P}_1^\alpha = D_1\mathcal{P}_0 D_1^\# & 0 \\ 0 & H_0\mathcal{P}_0 \end{pmatrix}$$

is the bosonic integral of motion = central charge of supersalgebra, $\ker \mathcal{P}_1^\alpha = \text{span}\,\{\xi^{-1}, \xi, \xi^3\}$.

One also can consider a bosonic integral $\mathcal{L}_2 = \sigma_3 \mathcal{L}_1$. It transforms mutually the first and second order supercharges:

$$[\mathcal{L}_2, Q_a] = 2i\mathcal{H}S_a\,, \qquad [\mathcal{L}_2, S_a] = -2i\mathcal{H}^2 Q_a\,.$$

Another example of superextended system

$$\mathcal{H} = \begin{pmatrix} H_1^{\alpha_2} & 0 \\ 0 & H_1^{\alpha_1} \end{pmatrix}, \qquad \alpha_1 > \alpha_2$$

Subsystems $H_1^{\alpha_1}$ and $H_1^{\alpha_2}$ can be intertwined by the second order differential operators $D_{\alpha_1} D_{\alpha_2}^{\#}$ and $D_{\alpha_2} D_{\alpha_1}^{\#}$ via the 'virtual' free particle system, $(D_{\alpha_1} D_{\alpha_2}^{\#}) H_1^{\alpha_2} = H_1^{\alpha_1} (D_{\alpha_1} D_{\alpha_2}^{\#})$, $(D_{\alpha_2} D_{\alpha_1}^{\#}) H_1^{\alpha_1} = H_1^{\alpha_2} (D_{\alpha_2} D_{\alpha_1}^{\#})$. However, there also exists the first order intertwiners, $D = \frac{d}{dx} + \mathcal{W}$, $D^{\#} = -\frac{d}{dx} + \mathcal{W}$, where $\mathcal{W} = \frac{1}{\xi_1} - \frac{1}{\xi_2} - \frac{1}{\xi_1 - \xi_2}$, $\xi_j = x + i\alpha_j$: $DH_1^{\alpha_1} = H_1^{\alpha_2} D$, $D^{\#} H_1^{\alpha_2} = H_1^{\alpha_1} D^{\#}$. They also satisfy the relations $D^{\#} D = H_1^{\alpha_1} - \Delta^2$, $DD^{\#} = H_1^{\alpha_2} - \Delta^2$, where $\Delta = (\alpha_1 - \alpha_2)^{-1}$.

Integrals:

$$Q_1 = \begin{pmatrix} 0 & D \\ D^\# & 0 \end{pmatrix}, \qquad S_1 = \begin{pmatrix} 0 & D_{\alpha_2} D^\#_{\alpha_1} \\ D_{\alpha_1} D^\#_{\alpha_2} & 0 \end{pmatrix},$$

$$\mathcal{L}_1 = \begin{pmatrix} \mathcal{P}^{\alpha_2} & 0 \\ 0 & \mathcal{P}^{\alpha_1} \end{pmatrix},$$

$Q_2 = \sigma_3 Q_1$, $S_2 = \sigma_3 s_1$, $\mathcal{L}_2 = \sigma_3 \mathcal{L}_1$. Nontrivial superalgebraic relations:

$$\{Q_a, Q_b\} = 2\delta_{ab}(\mathcal{H} - \Delta^2)\,, \qquad \{S_a, S_b\} = 2\delta_{ab}\mathcal{H}^2\,,$$

$$\{Q_a, S_b\} = 2\left(\epsilon_{ab}\mathcal{L}_1 + i\delta_{ab}\Delta\mathcal{H}\right)\,,$$

$$[\mathcal{L}_2, Q_a] = 2(i(\mathcal{H}-\Delta^2)S_a+\Delta\cdot\mathcal{H}Q_a)\,,\ [\mathcal{L}_2, S_a] = -2(i\mathcal{H}^2 Q_a+\Delta\cdot\mathcal{H}S_a)\,.$$

Exotic supersymmetry is in partially spontaneously broken phase: the doublet of bound states $\Psi_0^\pm = (D_{\alpha_2}1, \pm D_{\alpha_1}1)^t = (-\xi_2^{-1}, \mp\xi_1^{-1})^t$ of zero energy are not annihilated by the first order supercharges: $Q_1\Psi_0^\pm = \pm i\Delta\Psi_0^\pm$.

Superconformal symmetry of the system $\mathcal{H} = \mathrm{diag}(H_1^\alpha, H_0)$.

Integrals: $\mathcal{Q}_a$, $\mathcal{S}_a$, $\mathcal{L} = \mathcal{L}_1$, $\mathcal{D} = \mathrm{diag}\,(D_1^\alpha, D_0^\alpha)$,
$\mathcal{K} = \mathrm{diag}\,(K_1^\alpha, K_0^\alpha)$, $D_0^\alpha = D_0(x + i\alpha)$, $K_0^\alpha = K_0(x + i\alpha)$,
$D_0 = \frac{1}{4}\{G_0, \mathcal{P}_0\}$, $K_0 = G_0^2$, $G_0 = x - 2t\mathcal{P}_0$,

$$\mathcal{G} = \mathrm{diag}\,\left(G_1^\alpha,\, \tfrac{1}{2}\{G_0^\alpha, H_0\}\right)\,,\ \mathcal{V} = i\xi^2 - (\tfrac{d}{dx} + \xi^{-1})\mathcal{I} - 4t\mathcal{G} - 4t^2\mathcal{L}\,,$$

$$\mathcal{R} = \xi^3\mathcal{I} - 6t\mathcal{V} - 12t^2\mathcal{G} - 8t^3\mathcal{L}\,,$$

$$\mathcal{P}_- = \tfrac{1}{2}(1 - \sigma_3)\mathcal{P}_0\,, \qquad \mathcal{G}_- = \tfrac{1}{2}(1 - \sigma_3)G_0^\alpha\,,$$

$$\lambda_1 = \begin{pmatrix} 0 & i\xi \\ -i\xi & 0 \end{pmatrix} - 2t\mathcal{Q}_1\,, \qquad \lambda_2 = i\sigma_3\lambda_1\,,$$

$$\mu_1 = \begin{pmatrix} 0 & \xi\mathcal{P}_0 \\ \mathcal{P}_0\xi & 0 \end{pmatrix} - 2t\mathcal{S}_1\,, \qquad \mu_2 = i\sigma_3\mu_1\,,$$

$$\kappa_1 = \begin{pmatrix} 0 & \xi^2 \\ \xi^2 & 0 \end{pmatrix} - 4t\mu_1 - 4t^2\mathcal{S}_1\,, \qquad \kappa_2 = i\sigma_3\kappa_1\,.$$

Nonlinear extension of $osp(2|2)$ superalgebra with coefficients to be of order not higher than two in generators

Bosonic integrals
$\mathcal{L}$, $\mathcal{H}$, $\mathcal{G}$, $\mathcal{P}_-$, $\mathcal{I} = \text{diag}(1,1)$, $\Sigma = \sigma_3$, $\mathcal{D}$, $\mathcal{V}$, $\mathcal{G}_-$, $\mathcal{K}$, $\mathcal{R}$
are eigenstates of $-i\mathcal{D}$, $[-i\mathcal{D}, \mathcal{O}] = s_{\mathcal{O}}\mathcal{O}$, of the eigenvalues
$s_{\mathcal{O}} = (3/2, 1, 1/2, 1/2, 0, 0, 0, -1/2, -1, -1, -3/2)$

In the case of the system $\mathcal{H} = \text{diag}(H_1^{\alpha_2}, H_1^{\alpha_1})$, superconformal algebra is more complicated.

The number of generattors is the same, but no odd fermionic generator has a definite scaling dimension (is not eigenstate of the operartor $-i\mathcal{D}$)

$osp(2|2)$ is not contained as a sub-superalgebra.

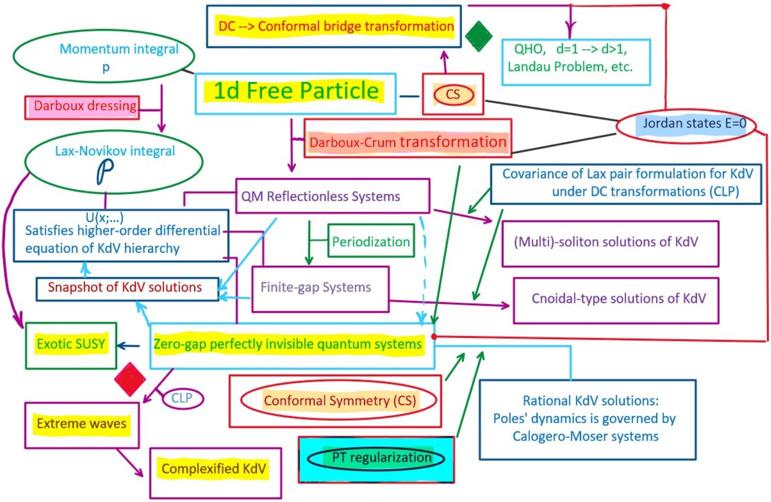
- The $\mathcal{PT}$-regularized superconformal mechanics systems in the phase of the unbroken exotic nonlinear $\mathcal{N} = 4$ super-Poincaré symmetry are described by nonlinearly super-extended Schrödinger algebra that involves the $osp(2|2)$ as a sub-superalgebra

- In the partially broken phase, the *scaling dimension* of all odd integrals is *indefinite*, and the $osp(2|2)$ *is not contained* as a sub-superalgebra.

- Additional bonus of the construction: extreme (rogue) wave solutions can be generated proceeding from the complexified KdV equation and higher equations of its hierarchy

- Some of the $\mathcal{PT}$-regularized conformal systems control stability properties of the kink-type solutions in the field-theoretical Liouville and $SU(3)$ conformal Toda systems

- Another peculiarity: Jordan states corresponding to zero energy play essential role in the construction



DC --> Conformal bridge transformation
QHO, d=1 --> d>1,
Landau Problem, etc.
Momentum integral
p
1d Free Particle
CS
Darboux dressing
Jordan states E=0
Darboux-Crum transformation
Lax-Novikov integral
𝒫
Covariance of Lax pair formulation for KdV
under DC transformations (CLP)
QM Reflectionless Systems
U(x;...)
Satisfies higher-order differential
equation of KdV hierarchy
Periodization
(Multi)-soliton solutions of KdV
Snapshot of KdV solutions
Finite-gap Systems
Cnoidal-type solutions of KdV
Exotic SUSY
Zero-gap perfectly invisible quantum systems
CLP
Conformal Symmetry (CS)
Rational KdV solutions:
Poles' dynamics is governed by
Calogero-Moser systems
Extreme waves
Complexified KdV
PT regularization

• Conformal bridge transformation: free particle $\longrightarrow$ QHO

Darboux transformation cannot make this job

$1d$ Free particle: $so(2,1) \cong sl(2,\mathbb{R})$ conformal symmetry:

$$\hat{H}_0 = -\tfrac{1}{2}\tfrac{d^2}{dx^2}\,, \qquad \hat{K} = \tfrac{1}{2}x^2\,, \qquad \hat{D} = -\tfrac{i}{2}\left(x\tfrac{d}{dx} + \tfrac{1}{2}\right)$$

$$[\hat{H}_0, \hat{D}] = -i\hat{H}_0\,, \qquad [\hat{H}_0, \hat{K}] = -2i\hat{D}\,, \qquad [\hat{K}, \hat{D}] = i\hat{K}$$

$$\hat{J}_0 = \frac{1}{2}(\hat{H}_0 + \hat{K}) = \hat{H}_{osc}\,, \quad \hat{J}_\pm = \hat{J}_1 \pm i\hat{J}_2 = -\frac{1}{2}(\hat{H}_0 - \hat{K} \pm i2\hat{D})$$

$$[\hat{J}_0, \hat{J}_\pm] = \pm\hat{J}_\pm\,, \qquad [\hat{J}_-, \hat{J}_+] = 2\hat{J}_0$$

Define non-unitary operator (it is *nonlocal*):

$$\hat{\mathfrak{S}} = e^{-\hat{K}} e^{i\ln 2\cdot\hat{D}} e^{\hat{H}_0} = \exp(\hat{J}_1 - \hat{J}_0)\cdot\exp(i\ln 2\cdot\hat{J}_2)\cdot\exp(\hat{J}_0 + \hat{J}_1)$$

It generates a canonical transformation identified as the fourth order root of the space reflection operator $\mathcal{P}$,

$$\hat{\mathfrak{S}} : (x, \hat{p}, \hat{a}^+, \hat{a}^-) \to (\hat{a}^+, -i\hat{a}^-, -i\hat{p}, x)$$
$$\hat{\mathfrak{S}}^2 : (x, \hat{p}, \hat{a}^+, \hat{a}^-) \to (-i\hat{p}, -ix, -\hat{a}^-, \hat{a}^+)$$
$$\hat{\mathfrak{S}}^4 : (x, \hat{p}, \hat{a}^+, \hat{a}^-) \to (-x, -\hat{p}, -\hat{a}^+, -\hat{a}^-)$$

$\Rightarrow \quad \hat{\mathfrak{S}}^8 = 1.$

It produces the non-unitary automorphism of $\mathfrak{sl}(2, \mathbb{R})$:

$$\hat{\mathfrak{S}} \hat{J}_0 \hat{\mathfrak{S}}^{-1} = i\hat{J}_2\,, \qquad \hat{\mathfrak{S}} \hat{J}_1 \hat{\mathfrak{S}}^{-1} = -\hat{J}_1\,, \qquad \hat{\mathfrak{S}} \hat{J}_2 \hat{\mathfrak{S}}^{-1} = -i\hat{J}_0$$

The action of the $\hat{\mathfrak{S}}^2$ on $\mathfrak{sl}(2, \mathbb{R})$ generators is a rotation by π about J_1: $\quad \hat{\mathfrak{S}}^2 : (\hat{J}_0, \hat{J}_1, \hat{J}_2) \to (-\hat{J}_0, \hat{J}_1, -\hat{J}_2)$

• $\hat{\mathfrak{S}}$ is $\mathcal{PT}$-invariant operator: $[\mathcal{PT}, \hat{\mathfrak{S}}] = 0$

• Since $\hat{\mathfrak{S}} i \hat{D} \hat{\mathfrak{S}}^{-1} = \hat{J}_0 = \hat{H}_{osc}$, it changes the form of dynamics in the sense of Dirac

• Classical analog of $\hat{\mathfrak{S}}$ generates complex canonical transformation in the phase space: $\widetilde{x} = a^+ \in \mathbb{C}$, $\widetilde{p} = -ia^- \in \mathbb{C}$

• Under complex conjugation, $\bar{\tilde{x}} = a^- = i\tilde{p}$, $\Rightarrow$ at the quantum level we pass over from the coordinate representation to representation in which $\hat{\tilde{x}} = \hat{a}^+$ acts as the operator of multiplication by $z \in \mathbb{C}$, $\hat{a}^+ \psi(z) = z\psi(z)$, while $i\hat{\tilde{p}} = \hat{a}^-$ acts as $\hat{a}^- \psi(z) = \frac{d}{dz}\psi(z)$

Replacing

$$(\psi_1, \psi_2) = \int_{-\infty}^{+\infty} \overline{\psi_1(x)}\psi_2(x)dx$$

by

$$(\psi_1, \psi_2) = \frac{1}{\pi}\int_{\mathbb{R}^2} \overline{\psi_1(z)}\psi_2(z)e^{-\bar{z}z}d^2z, \quad d^2z = d(\mathrm{Re}\, z)d(\mathrm{Im}\, z)\,,$$

$\Rightarrow$ we arrive at the Fock-Bargmann representation where $\hat{a}^+ = z$, $\hat{a}^- = \frac{d}{dz}$, $(\hat{a}^+)^\dagger = \hat{a}^-$ in correspondence with the classical relation $\overline{\tilde{x}} = a^- = i\tilde{p}$

In this representation

$$2i\widehat{\widetilde{D}} = \hat{H}_{\rm osc} = \left(z\frac{d}{dz} + \frac{1}{2}\right), \qquad \widehat{\widetilde{H}}_0 = -\frac{1}{2}\frac{d^2}{dz^2}, \qquad \widehat{\widetilde{K}} = \frac{1}{2}z^2$$

• ⇒ Transformed operators in the Fock-Bargmann representation can be obtained from the corresponding initial generators of conformal symmetry of the quantum free particle by a formal change of x to z.

• The change of the scalar product transmutes then the non-unitary similarity transformation into the *unitary transformation from the coordinate to the holomorphic representation* for the Heisenberg algebra in correspondence with the Neumann-Stone theorem

The conformal bridge generated by $\hat{\mathfrak{S}}$ transforms:

• Jordan states x^n ($x^0 = 1$ and x^1 are physical and nonphysical eigenstates) of $\hat{H}_0$, which are (formal) eigenstates of $i\hat{D}$ of eigenvalues $(n + 1/2)$, into eigenstates of $\hat{H}_{osc}$ of energies $E_n = n + 1/2$

• Plane waves eigenstates $e^{ikx} \longrightarrow$ coherent states of QHO

• Gaussian wave packets of the quantum free particle $\longrightarrow$ the squeezed states of the QHO

- Generallzations for other systems posessing conformal symmetry:
- $1d$ conformal mechanics $\longrightarrow$ de Alfaro, Fubini, Furlan model
- $2d$ free particle $\longrightarrow$ Landau problem
- conformal mechanics $\longrightarrow$ de Alfaro, Fubini, Furlan model in $3d$ monopole background
- free particle $\longrightarrow$ QHO in the cosmic string background (in preparation)

Thank you for your attention