Hamiltonian systems with balanced loss and gain: taming instabilities and the role of $\mathcal{PT}$-symmetry

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outline

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Defining systems with BLG

The flow in the position-velocity state-space preserves the volume, although individual particles are subjected to gain or loss.

- A system of $N$ particles with co-ordinates $x_i$

$$
\ddot{x}_i + \sum_{k=1}^{N} \eta_{ik}(x_1, x_2, \ldots, x_N) \dot{x}_i + G_i(x_1, x_2, \ldots, x_N) = 0
$$

- Position-velocity state-space volume $\mathcal{V} = \int \prod_{i=1}^{N} dx_i d\dot{x}_i$

$$
\frac{d\mathcal{V}}{dt} = 0 \Rightarrow \sum_{i=1}^{N} \eta_{ii}(x_1, x_2, \ldots, x_N) = 0 \Rightarrow Tr(\eta) = 0
$$
Hamiltonian system

- $\eta^T = -\eta$: No loss-gain terms, since $\eta_{ii} = 0 \ \forall \ i$
  - Admits a Hamiltonian for $G_i = -\frac{\partial \Gamma}{\partial x_i}$ with $\Gamma \equiv$potential
  - **Magnetic force**: Describes a system of interacting particles in presence of inhomogeneous magnetic field

- $\eta^T = \eta, \ Tr(\eta) = 0$: Is it a Hamiltonian system?
  - A few specific examples are known for $\eta_{ik} = (-1)^{i+1}\delta_{ik}$
  - Add BLG term $\eta_{ik} = (-1)^{i+1}\delta_{ik}$ to integrable systems like Calogero model, Toda chain, . . . , etc. No Hamiltonian yet!

- **Objective**: Hamiltonian formulation for general $\eta$ and $N$
  - **Advantages**: Application of canonical perturbation theory, canonical quantization scheme, geometric mechanics, KAM theory, tools for studying phase-transitions in configuration space, . . .
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Systems with Balanced Loss and Gain (BLG)

An Example: Hamiltonian system with BLG

\begin{align*}
\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x + \epsilon_2 y + \frac{g}{(x - y)^3} &= 0 \\
\ddot{y} - 2\gamma \dot{y} + \omega_0^2 y + \epsilon_1 x - \frac{g}{(x - y)^3} &= 0
\end{align*}

- $\epsilon_1 = \epsilon_2 = g = 0$: Bateman oscillator; No equilibrium state
  Ambient space is different from the target space
- $\epsilon_1 = \epsilon_2 \equiv \epsilon, g = 0$: Coupled oscillators with BLG
  Bender et. al., PRA 88, 062111 (2013)
- $\epsilon = -\omega_0^2, g \neq 0$: $A_2$-type Rational Calogero-Model with BLG
- RCM and coupled oscillator model
  - Admits periodic solution and quantum bound states
  - Target and ambient spaces are same
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Hamiltonian with BLG: Observations

The Hamiltonian in a coordinate system $z_\pm = x \pm y$

$$H = \left( P_{z_+} - \frac{\gamma}{2} z_- \right)^2 - \left( P_{z_-} + \frac{\gamma}{2} z_+ \right)^2 - \frac{\omega_0^2}{2} z_-^2 + \frac{g}{2 z_-^2}$$

- $H$ may be identified as defined in the background of a pseudo-Euclidean metric $M = \text{diag}(1, -1)$
- fictitious/analogous gauge potential: $A_\pm \equiv \pm \frac{\gamma}{2} z_\mp$
- Fictitious/analogous magnetic field: $\gamma$, gain-loss coefficient
- Imaginary scaling: $(z_- \rightarrow i \ z_-, P_{z_-} \rightarrow -i \ P_{z_-}) \Rightarrow H \rightarrow \tilde{H}$

$$\tilde{H} = \left( P_{z_+} - \frac{i \gamma}{2} z_- \right)^2 + \left( P_{z_-} + \frac{i \gamma}{2} z_+ \right)^2 + \frac{\omega_0^2}{2} z_-^2 - \frac{g}{2 z_-^2}$$

$\tilde{H}$ is defined in the background of a Euclidean metric with imaginary gauge potential
General Constructions

- Definitions, Notations etc.
  \[ X^T = (x_1, x_2, \ldots, x_N), \quad P^T = (p_1, p_2, \ldots, p_N), \]
  \[ F^T = (F_1, F_2, \ldots, F_N), \quad F_i \equiv F_i(x_1, x_2, \ldots, x_N) \]

- Generalized Momenta: \( \Pi = P + AF \), \( A \) is a constant matrix.
  \( A = AF \) may be interpreted as gauge potential

- \( H \) is defined in the constant background metric \( M \)
  \[ H = \Pi^T M \Pi + V(x_1, x_2, \ldots, x_N) \]
  \( M \) is a constant, non-singular, symmetric matrix

- Equations of Motion
  \[ \ddot{X} - 2D\dot{X} + 2M \frac{\partial V}{\partial X} = 0 \]
  \( D := MR, \quad R \equiv AJ - (AJ)^T, \quad [J]_{ij} \equiv \frac{\partial F_i}{\partial x_j} \)
Generic features

- \( R = AJ - (AJ)^T \Rightarrow R^T = -R \)
- \( D = MR \Rightarrow Tr(D) = 0: \) BLG condition satisfied automatically
- Choose \( D = D^T \Rightarrow \{M, R\} = 0, \{M, D\} = 0, \{R, D\} = 0 \)
- \( H \) in the background of a Pseudo-Euclidean metric
  - Orthogonal diagonalization of the symmetric matrix \( M \)
    \[ M_d = \hat{O}M\hat{O}^T = \text{diagonal}(\lambda_1, -\lambda_1, \lambda_2, -\lambda_2, \ldots, \lambda_m, -\lambda_m) \]
  - Orthogonal transformation of co-ordinates
    \[ \tilde{X} = \hat{O}X, \tilde{P} = \hat{O}P, \tilde{\Pi} = \hat{O}\Pi, \hat{O}^T\hat{O} = I_{2m} \]
- Hamiltonian in the transformed co-ordinate
  \[ H = \tilde{\Pi}^T M_d\tilde{\Pi} + V(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_N) \]
Malady & Remedy

- KE term is not definite, since $M$ is not definite.
- An indefinite $M$ is due to $\{M, R\} = 0$, $\{M, D\} = 0$
- Decompose $D$ in terms of symmetric and anti-symmetric matrices

$$D = \frac{1}{2} [M, R] + \frac{1}{2} \{M, R\}$$
$$\begin{align*}
D_S & (D_S^T = D_S) \\
D_A & (D_A^T = -D_A)
\end{align*}$$

- $M$ may be definite only if $D_A \neq 0$, i.e. $D^T \neq D$
- $D_A \neq 0$ introduces additional magnetic force in the system
- Lorentz interaction may be used to tame instabilities in BLG
- Electric force may be introduced via the potential $V$
Hamiltonian & Equations of motion

A particle in two dimensions is subjected to uniform magnetic field $B$ perpendicular to the plane and a constant electric field $E$ along the $x_1$ direction. Particle is subjected to BLG with the strength $\gamma$.

\[
\ddot{x}_1 + \gamma \dot{x}_1 - B \dot{x}_2 = E, \quad \ddot{x}_2 - \gamma \dot{x}_2 + B \dot{x}_2 = 0,
\]

- The Hamiltonian with $\Pi_i = P_i + \frac{1}{2} \epsilon_{ij} x_j$ and $\omega^2 = B^2 - \gamma^2$:

\[
H = \frac{B}{2} (\Pi_1^2 + \Pi_2^2) + \frac{\gamma}{2} \{\Pi_1, \Pi_2\} - \frac{E}{\omega^2} (Bx_1 - \gamma x_2),
\]

- Eigenvalues of $M = \frac{1}{2} (B I + \gamma \sigma_x)$ are $\lambda_{\pm} = \frac{1}{2} (B \pm \gamma)$

- Three distinct regions in the parameter space based on $\lambda_{\pm}$
  - Region-I ($B > \gamma$): $M$ is positive-definite
  - Region-II ($-\gamma < B < \gamma$): $M$ is indefinite
  - Region-III ($B < -\gamma$): $M$ is negative definite
Results in Region-I: Classical system

- $E = 0$: The particle moves in an elliptic orbit with reduced cyclotron frequency $\omega = \sqrt{B^2 - \gamma^2}$
- $E \neq 0$: The Hall current is not along $\hat{x}_2$, rather it makes an angle $\Phi_H = \tan^{-1}(\frac{B}{\gamma})$ with the direction of the external electric field $\vec{E} = E\hat{x}_1$. Similar results exist in plasma.
Results in Region-I: Quantum system

The degenerate ground-state wave function for $E = 0$:

$$\phi(x_1, x_2) = Z^m e^{-\frac{|\omega|}{8}|Z|^2}, \ m \in \mathbb{Z}^*$$

$$Z \equiv \xi x_1 - \xi^* x_2, \ \xi \equiv \sqrt{\frac{2}{|\omega|}} \left( \sqrt{|\lambda_+|} + i \sqrt{|\lambda_-|} \right)$$

- The most probable distribution $|\phi(x_1, x_2)|^2$ is centered around an ellipse instead of a circle.
- Hall effect ( $E \neq 0$ ): The probability current has non-vanishing components along both $\hat{x}_1$ and $\hat{x}_2$ directions.
- Pauli equation: $H_S = H + \frac{|\omega|}{2} \sigma_z$ for $E = 0$ is supersymmetric. Zeeman energy contains effective magnetic field

$$|\omega| = B \sqrt{1 - \left(\frac{\gamma}{B}\right)^2}$$
Coupled Duffing oscillator

\[ \ddot{x} + 2\Gamma \dot{x} + x + \beta y + \alpha x^3 = 0, \]
\[ \ddot{y} - 2\Gamma \dot{y} + y + \beta x + 3\alpha x^2 y = 0 \]
\[ H = (P_x + \Gamma y)(P_y - \Gamma x) + xy + \frac{\beta}{2}(x^2 + y^2) + \alpha x^3 y \]

- \( \beta = 0 \) corresponds to unforced damped Duffing oscillator in terms of \( x \). The \( y \) degree of freedom is unidirectionally coupled to \( x \) so that the system is Hamiltonian.
- \( \beta \neq 0 \): \( \beta y \) term acts as a forcing term in a nontrivial way.
- The nonlinear term breaks \( \mathcal{PT} \)-symmetry even for the general form of parity transformation in two dimensions.
- The system has five equilibrium points. Stability analysis by using Dirichlet theorem is inconclusive, since the second variation of \( H \) is not definite at the equilibrium points.
Perturbative & numerical analysis

- Linear stability analysis shows that three equilibrium points are stable.
- Perturbative analysis by using method of multiple time-scales is in conformity with linear stability analysis.
- Numerical analysis shows regular as well as chaotic behaviour.

Figure: Bifurcation diagrams for $\beta$ with $\Gamma = 0.01$ and $\alpha = .5$ with the initial conditions $x(0) = 0.01$, $y(0) = .02$, $\dot{x}(0) = .03$, $\dot{y}(0) = .04$.
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Non-$\mathcal{PT}$-symmetric Hamiltonian with BLG

$\beta < 1.01$: Regular dynamics

Figure: Regular solutions in the vicinity of the origin with the initial conditions $x(0) = .1, y(0) = 0.2, \dot{x}(0) = .03$ and $\dot{y}(0) = .04$. 

(a) $\alpha = 1, \beta = .5, \Gamma = .2$ 

(b) $\alpha = 1, \beta = .5, \Gamma = .2$
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Non-$\mathcal{PT}$-symmetric Hamiltonian with BLG

$\beta > 1.01$: Chaotic dynamics

\[ (a) \quad \Gamma = 0.01, \beta = 1.5, \alpha = 0.5 \]

\[ (b) \quad \Gamma = 0.01, \beta = 1.5, \alpha = 0.5 \]

**Figure:** Chaotic solutions with two sets of initial conditions
(a) $x(0) = 0.01, y(0) = 0.02, \dot{x}(0) = 0.03, \dot{y}(0) = 0.04$ (violet color) and (b) $x(0) = 0.01, y(0) = 0.02, \dot{x}(0) = 0.03, \dot{y}(0) = 0.025$ (green colour)
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Non-$\mathcal{PT}$-symmetric Hamiltonian with BLG

Poincaré section and Lyapunov exponent

(a) Poincaré section

(b) Lyapunov exponents

Figure: $\Gamma = 0.01, \beta = 1.5, \alpha = .5$
Initial condition $x(0) = .01, y(0) = 0.02, \dot{x}(0) = .03, \dot{y}(0) = .04$

- Lyapunov exponents: (.13248, .0015691, −.0016145, −.13244).
- Power-spectra and auto-correlation function show chaotic behaviour
Exactly solved Non-$\mathcal{PT}$-symmetric Dimer model with BLG

Multiple time-scale analysis of the coupled Duffing oscillator model for $\Gamma = \epsilon^2 \Gamma_0, \beta = \epsilon^2 \beta_0, \epsilon \ll 1$ produces the dimer model in the leading order of the perturbation with $T_2 = \epsilon^2 t$ and $A^T = (A_1, A_2)$

$$2i \frac{\partial A}{\partial T_2} + (2i \Gamma_0 \sigma_3 + \beta_0 \sigma_1) A + 3\alpha \left( \frac{|A_1|^2 A_1}{2|A_1|^2 B_1 + A_1^2 B_1^*} \right) = 0,$$

$A$ is the amplitude of the solutions of the Duffing oscillator model

- No $\mathcal{PT}$-symmetry for $\alpha \neq 0$
- Integrable and analytical periodic solutions are obtained
- Admits a stationary mode — $|A_1|, |B_1|$ are time-independent
- Non-$\mathcal{PT}$-symmetric Vector NLSE (VNLSE) may be obtained by adding a term $\frac{\partial^2 A}{\partial x^2}$ to the Dimer equation
- VNLSE admits periodic solutions for the moments $Z_a = 2 \int dx \, A^\dagger \sigma_a A$, $R = 2 \int dx \, A^\dagger A$
Summary

- **Taming the instability**: The kinetic energy term of a Hamiltonian system with BLG may be made positive-definite with an additional magnetic interaction leading to improved stability of the system.

- **Landau Hamiltonian with BLG**
  - The classical particle moves on an elliptic orbit with a cyclotron frequency that is less than that of the Landau system without BLG.
  - For the quantum particle, the most-probable probability density of the ground-state wave-function is centred around an ellipse.
  - Addition of Pauli term leads to a SUSY Hamiltonian with an effective magnetic field depending on the loss-gain parameter.

- **Hall effect**: The Hall current has non-vanishing components along the direction of the applied electric field. This is seen for both classical as well as quantum systems.
Role of $PT$-symmetry

- Non-$PT$-symmetric Hamiltonian systems with BLG are shown to admit periodic solutions
  - Examples include coupled Duffing oscillator model, an exactly solvable dimer model and a vector nonlinear Schrödinger equation
  - Coupled Duffing oscillator model provides an example of Hamiltonian chaos in systems with BLG.
  - Dimer model has a stationary mode

- $PT$ symmetry alone cannot determine whether or not a classical system with BLG admits periodic solutions

- Anti-linear symmetry may be substituted with the standard notion of $PT$-symmetry in quantum mechanics for explaining entirely real spectra of a non-hermitian system. In classical mechanics, such concepts are not used.

- A generalized criteria for predicting periodic solutions in classical systems with BLG is required
Coupled Duffing Oscillator & $\mathcal{PT}$-symmetry

(This part is added afterwards in response to questions during the presentation)

- Time-reversal symmetry in Classical Mechanics:
  $\mathcal{T} : t \rightarrow -t, P_x \rightarrow -P_x, P_y \rightarrow -P_y$
- $\mathcal{P}_D$: Parity transformation in $D$ dimensions with $\theta \in (0, 2\pi)$
  
  $\mathcal{P}_1 : \quad x \rightarrow -x, y \rightarrow -y, P_x \rightarrow -P_x, P_y \rightarrow -P_y$
  $\mathcal{P}_2 : \quad x \rightarrow x \cos \theta + y \sin \theta, y \rightarrow x \sin \theta - y \cos \theta,$
  $P_x \rightarrow P_x \cos \theta + P_y \sin \theta, P_y \rightarrow P_x \sin \theta - P_y \cos \theta$

- $H$ may be interpreted as a system of (a) two particles in one dimension or (b) one particle in two dimensions.
- (a) Term linear in $\Gamma$ in $H$ is non-invariant under $\mathcal{P}_1 \mathcal{T}$
- (b) $\alpha = 0$: $H$ is invariant under $\mathcal{P}_2 \mathcal{T}$ for $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ only
  $\alpha \neq 0$: $H$ is non-invariant under $\mathcal{P}_2 \mathcal{T}$ for any value of $\theta$

Nonlinear interaction breaks $\mathcal{PT}$ symmetry
Lorentz invariance, $\mathcal{PT}$-symmetry & Beyond

- There are no signatures of violation of Lorentz invariance in nature so far.
- The $CPT$-norm is introduced for non-relativistic $\mathcal{PT}$ symmetric quantum systems with the motivation that at a more fundamental level, it may correspond to $CPT$ theorem that is based on Lorentz invariant, hermitian quantum field theory.
- Both $\mathcal{P}_1$ and $\mathcal{P}_2$ correspond to improper Lorentz transformations in respective space dimensions.
- The parity operation may be defined as a mathematical entity in terms of nonlinear transformations. This is not possible within the framework of Lorentz transformations, since it is a linear transformation.
- Introducing parity operation involving nonlinear transformations for $\mathcal{PT}$-symmetric classical system amounts to abandoning Lorentz invariance at a more fundamental level.
A particular choice for $N = 2m$

- $M = I_m \otimes \sigma_x + \alpha^2 I_{2m}$, $A = \frac{-i\gamma}{2} I_m \otimes \sigma_y$, $\alpha \in \mathbb{R}$
- Assumption: $F_{2i-1} \equiv F_{2i-1}(x_{2i-1}, x_{2i})$, $F_{2i} \equiv F_{2i}(x_{2i-1}, x_{2i})$
- $J$ has the expression: $J = \sum_{i=1}^{m} U_i^{(m)} \otimes V_i^{(2)}$

$$\begin{bmatrix} U_a^{(m)} \end{bmatrix}_{ij} \equiv \delta_{ia} \delta_{ja}, \quad V_a^{(2)} \equiv \begin{pmatrix} \frac{\partial F_{2a-1}}{\partial x_{2a-1}} & \frac{\partial F_{2a-1}}{\partial x_{2a}} \\ \frac{\partial F_{2a}}{\partial x_{2a-1}} & \frac{\partial F_{2a}}{\partial x_{2a}} \end{pmatrix}$$

- $Q_a(x_{2a-1}, x_{2a}) \equiv \text{Trace}(V_a^{(2)})$, $[\chi_m]_{ij} \equiv \frac{1}{2} \delta_{ij} Q_i(x_1, x_2, \ldots, x_N)$

$$R = \frac{\gamma}{2} \sum_{i=1}^{m} U_i^{(m)} \otimes \begin{pmatrix} 0 & -Q_i(x_{2i-1}, x_{2i}) \\ Q_i(x_{2i-1}, x_{2i}) & 0 \end{pmatrix}$$

- $D = \gamma \chi_m \otimes \sigma_z + \alpha^2 R$
- $M$ is positive-definite for $\alpha^2 > 1$
Pseudo-Euclidean metric

$\hat{O} = \frac{1}{\sqrt{2}} \left[ I_m \otimes (\sigma_x + \sigma_z) \right]$ diagonalizes $M$ and generates the Co-ordinate transformation

$z_i^\pm = \frac{1}{\sqrt{2}}(x_{2i-1} \pm x_{2i}), \; P_{z_i^\pm} = \pm \frac{1}{2} (\dot{z}_i^\pm - \gamma F_i^\mp)$

$F_i^\pm = \frac{1}{\sqrt{2}} (F_{2i-1} \pm F_{2i}), \; F_i^\pm \equiv F_i^\pm(z_i^+, z_i^-)$

$H$ describes a system of $m$ particles on a Pseudo-Euclidean plane interacting with each other through $V$

$H = \sum_{i=1}^{m} \left[ \left( P_{z_i^+} + \frac{\gamma}{2} F_i^- \right)^2 - \left( P_{z_i^-} - \frac{\gamma}{2} F_i^+ \right)^2 \right] + V$

The $i$'th particle is subjected to ‘fictitious magnetic field’ $Q_i$

$Q_i = \frac{\partial F_i^+}{\partial z_i^+} + \frac{\partial F_i^-}{\partial z_i^-}$


