PT Phase Transition: QM to QFT

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Based on our works

- A PT-invariant potential with complex QES eigenvalues, A Khare & B P Mandal, PLA (2000)
- PT phase transition in higher-dimensional quantum systems, B P Mandal, B K Mourya, R K Yadav PLA(2013)
- PT phase transition in a $(2+1)$-d relativistic system, B P Mandal, B K Mourya, K Ali, & A Ghatak AOP(2015)
- Deconfinement to confinement as PT phase transition, H Rawal & B P Mandal NPB (2019)

- If a theory has anti-linear symmetry, then there are two different phases. One side we can have simultaneous eigenfunctions of Hamiltonian and anti-linear operator, and consistent quantum theory. (Unbroken Phase)
- In the other phase symmetry is broken spontaneously and we will have difficulties in obtaining a consistent theory.
- With change of certain parameters in the theory the system may transit from one phase to another. Very rich physics is associate with this transition.
- In this talk, we will discuss PT phase transition in one dimension, higher dimension and in relativistic quantum mechanical systems, mainly with the help of certain models. Finally we will discuss PT phase transition in SU(N) gauge field theory.

1-d: Complex QES:DSHG

- We consider

$$H = p^2 - (\zeta \cosh 2x - iM)^2$$

  is PT symmetric Non-Hermitian system. ζ M are real.
- Where $P: \quad x \to a - x, \quad p \to -p$
  $T: i \to -i, \quad p \to -p; \quad x \to x$
- This is a nice example of PT symmetric Non-Hermitian QES system. For real positive integer value of $M = n$, n levels can be exactly solved.
- For M=odd, we can find a $\zeta \leq \zeta_c$ for which entire QES spectrum is real and QES wavefunctions are eigenstates of the PT.
- For M=even, For the entire range of the parameter ζ QES eigenvalues are in complex conjugate pairs and the system is always in PT broken phase.

- Anti-isospectral transformation Or duality transformation: If under $x \longrightarrow ix \equiv y$ the potential $v(x) \longrightarrow \bar{v}(y)$, and if the potential $v(x)$ has M QES levels with energy eigenvalue and eigenfunctions E_k $(k = 0, 1, 2 \cdots M-1)$ and $\psi_k(x)$ respectively then the energy eigenvalues of $\bar{v}(y)$ are given by

$$\bar{E}_k = -E_{M-1-k}, \quad \bar{\psi}_k(y) = \psi_{M-1-k}(ix) \tag{1}$$

- Under anti-isopectral transformation, $x \longrightarrow ix \equiv \theta$ DSHG potenal changes to DSG potential given by

$$V(\theta) = (\zeta \cos 2\theta - iM)^2.$$

  The levels for DSG system are obtained.
- The Hamiltonian for DSHG and DSG are expressed in terms of generators $SL(2, R)$ atmost in quadratic.

Anisotropic Oscillator with imaginary Coupling

- We consider a non-Hermitian anisotropic oscillator in two dimension.

$$
\begin{aligned}
H &= \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\omega_x^2 x^2 + \frac{1}{2}m\omega_y^2 y^2 + i\lambda xy, \\
&= H_0 + H_{nh} \quad \lambda \text{ is real and } \omega_x \neq \omega_y
\end{aligned}
$$

- The Hamiltonian is invariant under $P_1 T$ and $P_2 T$ as

$$P_1 T(i\lambda xy) = P_2 T(i\lambda xy) = i\lambda xy$$

- Parity in 2-d is,

$$P_1 : x' = -x, \quad y' = y \tag{2}$$

$$P_2 : x' = x, \quad y' = -y \tag{3}$$

Parity in 2d

- Both of these forms P_1 and P_2 are equivalent and one can use either of these while checking PT symmetry of a non-Hermitian system in two dimensions. In two dimension parity transformation has also been realized as

$$P_3 : x \longrightarrow y, \;\; y \longrightarrow x \tag{4}$$

leading to the transformation matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5}$$

- P_3 is not considered here as H_0 is not invariant under it.

Decoupling the model

- This Hamiltonian can be decoupled by making a coordinate transformation $(x, y) \rightarrow (X, Y)$ as-

$$H = \frac{P_X^2}{2m} + \frac{P_Y^2}{2m} + \frac{1}{2} m C_1^2 X^2 + \frac{1}{2} m C_2^2 Y^2 \tag{6}$$

- where,

$$X = \sqrt{\frac{1+k}{2}} x - \sqrt{\frac{1-k}{2}} y; \qquad Y = \sqrt{\frac{1-k}{2}} x + \sqrt{\frac{1+k}{2}} y; \tag{7}$$

and

$$C_1^2 = \frac{1}{2} \left[\omega_+^2 - \frac{\omega_-^2}{k} \right]; \qquad C_2^2 = \frac{1}{2} \left[\omega_+^2 + \frac{\omega_-^2}{k} \right]; \tag{8}$$

$$\omega_+^2 \equiv \omega_x^2 + \omega_y^2; \qquad \omega_-^2 \equiv \omega_y^2 - \omega_x^2; \qquad k^{-1} \equiv \sqrt{1 - \frac{4\lambda^2}{m^2 \omega_-^4}}$$

Solutions

- $$E_{n_1,n_2} = (n_1 + \frac{1}{2})\hbar C_1 + (n_2 + \frac{1}{2})\hbar C_2 \qquad (9)$$

- $$\psi_{n_1,n_2}(X,Y) = N \exp\left[-(\frac{\alpha_1^2 X^2}{2} + \frac{\alpha_2^2 Y^2}{2})\right] H_{n_1}(\alpha_1 X) H_{n_2}(\alpha_2 Y) \qquad (10)$$

  where, $\alpha_1^2 = \frac{mC_1}{\hbar}$ and $\alpha_2^2 = \frac{mC_2}{\hbar}$.

- If $|\lambda| \leq |\frac{m\omega^2}{2}|$. then k is real and $k \geq 1$

$$C_1^2 = \frac{1}{2}[\omega_x^2(1 + \frac{1}{k}) + \omega_y^2(1 - \frac{1}{k})] > 0,$$
$$C_2^2 = \frac{1}{2}[\omega_x^2(1 - \frac{1}{k}) + \omega_y^2(1 + \frac{1}{k})] > 0, \; as \; k \geq 1$$

Unbroken PT phase

- The entire spectrum is real

$$E_{n_1,n_2} = (n_1 + \frac{1}{2})\hbar C_1 + (n_2 + \frac{1}{2})\hbar C_2 \qquad (11)$$

- The wave function

$$\psi_{n_1,n_2}(x,y) = N \exp\left[-\frac{m}{2\hbar}[(C_1 + C_2)(x^2 + y^2) + (C_2 - C_1)2i\lambda kxy\right]$$

$$H_{n_1}[\alpha_1(\sqrt{\frac{k+1}{2}}x - i\sqrt{\frac{k-1}{2}}y)]H_{n_2}[\alpha_2(\sqrt{\frac{k-1}{2}}x + i\sqrt{\frac{k+1}{2}}y)]$$

- Under PT transformation

$$P_1 T\psi_{n_1,n_2}(x,y) = (-1)^{n_1+n_2}\psi_{n_1,n_2}(x,y) = \pm\psi_{n_1,n_2}(x,y), \qquad (12)$$

$$P_2 T\psi_{n_1,n_2}(x,y) = +\psi_{n_1,n_2}(x,y) \qquad (13)$$

as n_1, n_2 are zero or positive integers.

- PT is unbroken as long as $|\lambda| \leq |\frac{m\omega_-^2}{2}| = |\frac{m}{2}(\omega_x^2 - \omega_y^2)|$

Broken PT Phase

- For $|\lambda| > |\frac{m\omega_-^2}{2}|$, k is imaginary and hence $P_i T\psi_{n_1,n_2}(x,y) \neq \pm\psi_{n_1,n_2}(x,y)$ for $i = 1,2$
- the spectrum in this situation is

$$E_{n_1,n_2} = (n_1 + \frac{1}{2})\hbar(A - iB) + (n_2 + \frac{1}{2})\hbar(A + iB) \qquad (14)$$

- A and B are real and can be given as

$$A^2 = \frac{1}{2}\left[\omega_+^2 + \sqrt{\frac{\omega_+^4 + \omega_-^4}{k_1^2}}\right]; \qquad B^2 = \frac{1}{2}\left[-\omega_+^2 + \sqrt{\frac{\omega_+^4 + \omega_-^4}{k_1^2}}\right] \qquad (15)$$

  $k_1(\equiv -ik)$ is also real.
- It is clear from equation that E_{n_1,n_2} and E_{n_2,n_1} are complex conjugate to each other.
- On the other hand energy eigenvalues are real when $n_1 = n_2$. Still the system is in broken phase.

Comments

- The critical value of the coupling $\lambda_c (= \frac{m\omega_-^2}{2})$ depends on the anisotropy of the system. If the system is more anisotropic the span of the PT unbroken phase is wider.
- When the system becomes isotropic, i.e. $\omega_x = \omega_y$, i.e. $\lambda_c = 0$,the system will always lie in the broken PT phase and it will not be possible to have entire spectrum real for any condition on the parameters.
- Rotational symmetry which leads to isotropy of original Hermitian system H_0 seems to be complementary to unbroken PT symmetry of the PT symmetric non-Hermitian system H.
- As long as the original system H_0 has rotational invariance, H can not have unbroken PT symmetry. The moment rotational symmetry of H_0 breaks the non-Hermitian system becomes capable of going through a PT phase transition.

PT Phase transition in 3d

- 3-d isotropic SHO in external imaginary magnetic field (iB) can be written as

$$H = \frac{1}{2m}(\vec{p} - \frac{iq\vec{A}}{c})^2 + \frac{1}{2}m\omega^2(x^2 + y^2 + z^2) + i\vec{\mu}_l.\vec{B} \quad (16)$$

where q is the charge of the oscillator and imaginary magnetic field is assumed z direction.

- The vector potential $\vec{A} = \{-\frac{By}{2}, \frac{Bx}{2}, 0\}$ in symmetric gauge the Hamiltonian can written as

$$H = \frac{1}{2m}(p_x + \frac{iqyB}{2c})^2 + \frac{1}{2m}(p_y - \frac{iqxB}{2c})^2 + \frac{1}{2m}(p_z)^2 + \frac{1}{2}m\omega^2(x^2 + y^2 + z^2) + i\mu_{lz}B \quad (17)$$

- This non-Hermitian Hamiltonian is PT invariant.

- The Hamiltonian reduced to a 3d anisotropic oscillator

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + \frac{1}{2}m\omega_1^2(x^2+y^2) + \frac{1}{2}m\omega^2 z^2 \quad (18)$$

$$\text{where}, \omega_1^2 = \omega^2 - \frac{q^2B^2}{4m^2c^2} = \omega^2 - \frac{\omega_c^2}{4} \quad (19)$$

$\omega_c = \frac{qB}{mc}$ is usual cyclotron frequency.

- The energy eigenvalue and eigenfunction for this system are given as,

$$E_{n_x n_y n_z} = (n_x + n_y + 1)\hbar\omega_1 + (n_z + \frac{1}{2})\hbar\omega \quad (20)$$

$$\psi_{n_x n_y n_z} = \exp\left[-\frac{\alpha^2}{2}(x^2+y^2) + \frac{\alpha_1^2 z^2}{2}\right] H_{n_x}(\alpha x) H_{n_y}(\alpha y) H_{n_z}(\alpha_1 z)$$

$\alpha^2 = \frac{m\omega_1}{\hbar}$, $\alpha_1^2 = \frac{m\omega}{\hbar}$.

PT Phase Transition

- If the magnetic field is sufficiently weak, $B \leq \frac{2m\omega c}{q}$ or for a fixed magnetic field, oscillator frequency $\omega \geq \frac{\omega_c}{2}$, then ω_1 is real and hence the entire spectrum is real. In this case it is straight forward to check $PT\psi_{n_x n_y n_z}(x,y,z) = (-1)^{n_x+n_y+n_z}\psi_{n_x n_y n_z}(x,y,z)$, indicating the system is in unbroken PT phase.
- However if the strength of the magnetic field exceeds a critical value $B > B_c = \frac{2m\omega c}{q}$ or the oscillator frequency is less then half of the cyclotron frequency for fixed magnetic field i.e $\omega \leq \frac{\omega_c}{2}$, then $\omega_1 = \pm\sqrt{\omega^2 - \frac{q^2B^2}{4m^2c^2}}$, becomes complex i.e $\omega_1 \equiv \pm i\tilde{\omega}$ where $\tilde{\omega} \equiv \sqrt{-\omega^2 + \frac{q^2B^2}{4m^2c^2}}$ is real. We have pairs of complex conjugate eigenvalues given as

$$E_{n_x n_y n_z} = \pm i(n_x + n_y + 1)\hbar\tilde{\omega} + (n_z + \frac{1}{2})\hbar\omega \qquad (21)$$

Remarks

- The system is in broken phase of PT, this is because ω_1 is imaginary and under PT it changes to $-\omega_1$ hence $PT\psi_{n_x n_y n_z}(x, y, z) \neq \pm\psi_{n_x n_y n_z}(x, y, z)$. The system is PT unbroken phase if $0 < B \leq B_c$ for fixed oscillator frequency ω and PT phase transition occurs as the strength of magnetic field exceeds the critical value.

- Dirac Oscillator in presence of magnetic field with imaginary Rasbha interaction

$$H = v_f[\vec{\sigma}.(\vec{\Pi} - iK_1\vec{r}\beta) + i\lambda(\vec{\sigma}\times\vec{\Pi}).\hat{z}, \qquad (22)$$

where $\vec{\Pi} = \vec{p} + \frac{e\vec{A}}{c}$, and K_1, λ are real constants and v_f is Fermi velocity.

- $B = B_0\hat{k}$, The Hamiltonian in the symmetric gauge is

$$H = v_f(\sigma_x\Pi_x + \sigma_y\Pi_y) - iK_1v_f(\sigma_x x + \sigma_y y)\beta + i\lambda(\sigma_x\Pi_y - \sigma_y\Pi_x)$$

where, $\Pi_x = p_x - \frac{eB_0 y}{2c}$ and $\Pi_y = p_y + \frac{eB_0 x}{2c}$.

- Parity transformation,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$$

- $$P_1 : x \longrightarrow -x, \quad y \longrightarrow y, \quad p_x \longrightarrow -p_x, \quad p_y \longrightarrow p_y. \quad (23)$$

  $$P_2 : x \longrightarrow x, \quad y \longrightarrow -y, \quad p_x \longrightarrow p_x, \quad p_y \longrightarrow -p_y. \quad (24)$$

- Under parity transformations Dirac wave functions transform

  $$\begin{aligned} P_1\psi(x,y,t) &= \sigma_y\psi(-x,y,t) \\ P_2\psi(x,y,t) &= \sigma_x\psi(x,-y,t) \end{aligned} \quad (25)$$

  such that Dirac equation remains invariant.

- The time reversal transformation $(i \longrightarrow -i, \quad p_x \longrightarrow -p_x, \quad p_y \longrightarrow -p_y)$ in (2+1) d Dirac theory is defined as $T = i\sigma_y\tilde{K}$, where $\tilde{K}$ is complex conjugation operation such that, $T\psi(x,y,t) = i\sigma_y\tilde{K}\psi(x,y,-t) = i\sigma_y\psi^\star(x,y,-t)$.

Two Valleys

- The system is Non-Hermitian.

$$\begin{aligned} H^\dagger &= v_f(\sigma_x \Pi_x + \sigma_y \Pi_y) - iK_1 v_f(\sigma_x x + \sigma_y y)\beta \\ &- i\lambda(\sigma_x \Pi_y - \sigma_y \Pi_x) \qquad \neq H \end{aligned}$$

- But invariant under both $P_1 T$ and $P_2 T$ as

$$\begin{aligned} P_1 TH(P_1 T)^{-1} &= H \\ P_2 TH(P_2 T)^{-1} &= H \end{aligned}$$

- The time reversal of the Hamiltonian describes the other valley

$$\begin{aligned} \tilde{H} &= THT^{-1} = v_f(\sigma_x \tilde{\Pi}_x + \sigma_y \tilde{\Pi}_y) - iK_1 v_f(\sigma_x x + \sigma_y y)\beta \\ &- i\lambda(\sigma_x \tilde{\Pi}_y - \sigma_y \tilde{\Pi}_x) \end{aligned} \tag{26}$$

where $\tilde{\Pi}_x = p_x + \frac{eB_0 y}{2c}$ and $\tilde{\Pi}_y = p_y - \frac{eB_0 x}{2c}$.

- Note: $\tilde{H} \neq \tilde{H}^\dagger$ and $[\tilde{H}, P_1 T] = [\tilde{H}, P_2 T] = 0$.

Solutions

- Solutions of DE for both the valleys show the PT phase transition and mass gap generation .
- The Hamiltonians H and $\tilde{H}$ on a complex plane $z = (x + iy)$ are,

$$H = \begin{pmatrix} 0 & A\Pi_z + iC_1\bar{z} \\ B\Pi_{\bar{z}} + iC_2 z & 0 \end{pmatrix} \quad , \quad \tilde{H} = \begin{pmatrix} 0 & B\Pi_z + iC_2\bar{z} \\ A\Pi_{\bar{z}} + iC_1 z & 0 \end{pmatrix} \tag{27}$$

  where, $A = 2(v_f - \lambda)$, $B = 2(v_f + \lambda)$, $C_1 = K_1 v_f - (v_f - \lambda)\frac{B_0 e}{2c}$ and $C_2 = -K_1 v_f + (v_f + \lambda)\frac{B_0 e}{2c}$ are constants.

- The canonical conjugate momenta Π_z $\Pi_{\bar{z}}$ are $\Pi_z = -i\hbar\frac{d}{dz} = \frac{1}{2}(p_x - ip_y)$, $\Pi_{\bar{z}} = -i\hbar\frac{d}{d\bar{z}} = \frac{1}{2}(p_x + ip_y)$

  $[\bar{z}, \Pi_{\bar{z}}] = i\hbar = [z, \Pi_z]$; $[z, \Pi_{\bar{z}}] = 0$; $[\bar{z}, \Pi_z] = 0$; $[\Pi_{\bar{z}}, \Pi_z] = 0$

Solutions

- We assume a solution of the corresponding Dirac equation in the two components form as,

$$\psi = \begin{pmatrix} \phi \\ i\chi \end{pmatrix} \tag{28}$$

- Dirac equation is then written in components form as

$$(A\Pi_z + iC_1\bar{z})(B\Pi_{\bar{z}} + iC_2 z)\phi = E^2\phi \tag{29}$$

$$(B\Pi_{\bar{z}} + iC_2 z)(A\Pi_z + iC_1\bar{z})\chi = E^2\chi \tag{30}$$

- Now we look for a solution of the kind

$$\begin{aligned} \phi &= \xi(z,\bar{z})e^{d_1 z\bar{z}} \\ \chi &= \eta(z,\bar{z})e^{d_1 z\bar{z}} \end{aligned} \tag{31}$$

Solutions

- Substituting these in DEs and comparing the coefficient of $(z\bar{z})$ in both sides we obtain $d_1 = \frac{C_1}{A\hbar}$ or $\frac{C_2}{B\hbar}$. The general solutions for the n^{th}

$$\psi_n^I(z,\bar{z},t) = \begin{pmatrix} \xi_n \\ i\eta_n \end{pmatrix} e^{\frac{C_1}{A\hbar}z\bar{z}} e^{-\frac{i}{\hbar}E_n t} = a_n^I \begin{pmatrix} z^n \\ iz^{n+1} \end{pmatrix} e^{\frac{C_1}{A\hbar}z\bar{z}} e^{-\frac{i}{\hbar}E_n t}$$

$${E^I}_n^2 = (n+1)(AC_2 - BC_1)$$

- In exactly similar fashion we obtain the solution corresponding to $d_1 = \frac{C_2}{B\hbar}$ as,

$$\psi_n^{II}(z,\bar{z},t) = a_n^{II} \begin{pmatrix} \bar{z}^{n+1} \\ i\bar{z}^n \end{pmatrix} e^{\frac{C_2}{B\hbar}z\bar{z}} e^{-\frac{i}{\hbar}\tilde{E}_n t},$$

$${E^{II}}_n^2 = (n+1)(BC_1 - AC_2) = -{E^I}_n^2$$

Solution for other valley

- The solution corresponding to the DE for $\tilde{H}$ can be found using analogy,

$$H \xrightarrow[\mathrm{C_1 \leftrightarrow C_2}]{\mathrm{A \leftrightarrow B}} \tilde{H}$$

- Interchanging $A \leftrightarrow B, C_1 \leftrightarrow C_2$ is equivalent to $\lambda \to -\lambda, K_1 \to -K_1$ and $B_0 \to -B_0$.
- The solutions for the systems $\tilde{H}$ are given as,

$$\text{For } \tilde{d}_1 = \frac{C_1}{A\hbar}, \quad \tilde{\psi}_n^I(z, \bar{z}, t) = \tilde{a}_n^I \begin{pmatrix} \bar{z}^{n+1} \\ i\bar{z}^n \end{pmatrix} e^{\frac{C_1}{A\hbar} z\bar{z}} e^{-\frac{i}{\hbar} E_n t} \quad (32)$$

$$\text{For } \tilde{d}_1 = \frac{C_2}{B\hbar}, \quad \tilde{\psi}_n^{II}(z, \bar{z}, t) = \tilde{a}_n^{II} \begin{pmatrix} z^n \\ iz^{n+1} \end{pmatrix} e^{\frac{C_2}{B\hbar} z\bar{z}} e^{-\frac{i}{\hbar} \tilde{E}_n t} \quad (33)$$

where $\tilde{E^I}_n^2 = (n+1)(AC_2 - BC_1) = -{E^{II}}_n^2$. .

Mass Gap

- The energy eigenvalues for the solutions corresponding to $d_1 = \tilde{d}_1 = \frac{C_1}{A\hbar}$ in both the valleys are,

$$E_n = \pm\sqrt{(n+1)\left[2(v_f^2 - \lambda^2)\frac{B_0 e\hbar}{c} - 4K_1 v_f^2 \hbar\right]} \tag{34}$$

- This indicates a mass gap

$$\Delta_0 = \sqrt{2(v_f^2 - \lambda^2)\frac{B_0 e\hbar}{c} - 4K_1 v_f^2 \hbar} \tag{35}$$

is generated between the positive and negative energy solutions due to the interactions present in this theory.

$P_1 T$ and $P_2 T$ Phase transition

- The energy eigenvalues are real for these solutions when

$$\text{(i) } \lambda^2 \leq v_f^2 \left(1 - \frac{2K_1 c}{B_0 e}\right) \equiv \lambda_c^2 \quad \text{OR}$$

$$\text{(ii) } B_0 > 2K_1 \frac{v_f^2 c}{(v_f^2 - \lambda^2) e} \equiv B_0^c \tag{36}$$

- As long as E_n is real ψ_n^I and $\tilde{\psi}_n^I$ are an eigenstate of both $P_1 T$ and $P_2 T$
- Similar interpretation can also be made for the case $d_1 = \tilde{d}_1 = \frac{C_2}{B\hbar}$

-
$$P_1T\psi_n^I = \sigma_y i\sigma_y \begin{pmatrix} \xi_n(-z,-\bar{z}) \\ -i\eta_n(-z,-\bar{z}) \end{pmatrix} e^{\frac{C_1}{A\hbar}z\bar{z}} e^{-\frac{i}{\hbar}E_n t}, \;\; P_1Tz = -z$$
$$= \;\; i(-1)^n \begin{pmatrix} \xi_n \\ i\eta_n \end{pmatrix} e^{\frac{C_1}{A\hbar}z\bar{z}} e^{-\frac{i}{\hbar}E_n t} = i(-1)^n \psi_n^I(z,\bar{z},t) \tag{37}$$

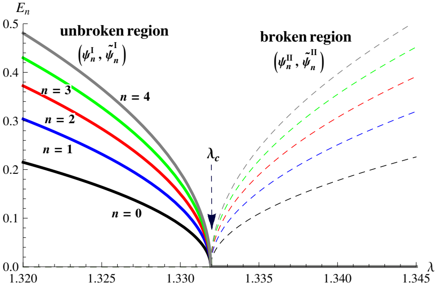
- Similarly,

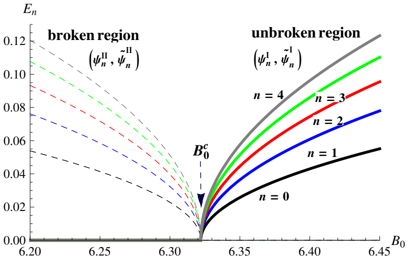
$$P_2T\psi_n^I \;\; = \;\; \sigma_x i\sigma_y \begin{pmatrix} \xi_n(z,\bar{z}) \\ -i\eta_n(z,\bar{z}) \end{pmatrix} e^{\frac{C_1}{A\hbar}z\bar{z}} e^{-\frac{i}{\hbar}E_n t}, \quad \text{as } P_2Tz = z$$
$$= \;\; -\begin{pmatrix} \xi_n \\ i\eta_n \end{pmatrix} e^{\frac{C_1}{A\hbar}z\bar{z}} e^{-\frac{i}{\hbar}E_n t} = -\psi_n^I(z,\bar{z},t) \tag{38}$$

- For the other valley as,

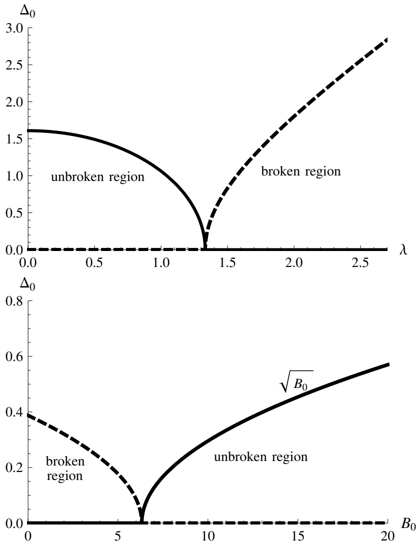
$$\begin{aligned} P_1T\tilde{\psi}_n^I(z,\bar{z},t) &= -i(-1)^n\tilde{\psi}_n^I(z,\bar{z},t) \\ P_2T\tilde{\psi}_n^I(z,\bar{z},t) &= -\tilde{\psi}_n^I(z,\bar{z},t) \end{aligned} \tag{39}$$

Phase transition





Mass Gap in unbroken Phase



Non-Hermitian Scalar Theory:Toy Model

- First we consider non-Hermitian theory of charged scalars

$$L = \partial_\mu \phi_1^* \partial^\mu \phi_1 + \partial_\mu \phi_2^* \partial^\mu \phi_2 + [\phi_1^* \quad \phi_2^*] M^2 \left[\begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right]$$

- where

$$M^2 = \begin{pmatrix} m_1^2 & \mu^2 \\ -\mu^2 & m_2^2. \end{pmatrix}$$

We will be interested for $m_1^2, m_2^2, \mu^2 \geq 0$. The mass matrix M^2 is not Hermitian.

Non-Hermitian Scalar Theory:Toy Model

- Discrete symmetries become easier when the doublet of two fields is defined as

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

- The parity and time reversal are defined as follows

$$\Phi \xrightarrow{\mathrm{P}} P\Phi$$
$$\Phi \xrightarrow{\mathrm{T}} T\Phi^*$$

P and T now are 2×2 matrices

Non-Hermitian Scalar Theory

- The parity transformation in $\mathbb{R}^2$, is $x \to x$ and $y \to -y$. This suggests in $\mathbb{R}^2$ the field ϕ_1 transforms as a scalar and the other, ϕ_2 transforms as a pseudo scalar. P in matrix form

$$P = \begin{bmatrix} 1 & 0 \\ 0 & -1. \end{bmatrix}$$

- The choice for the time reversal T under which the Lagrangian is PT-invariant, $\mathrm{T} = \mathbf{1}_2$.
- As long as symmetry is concern roles of P and T can be interchanged. However in order to interpret this PT-symmetric theory in terms of a coupled system with gain and loss, one should take $\mathrm{T} = \mathbf{1}_2$.

- The theory remains in the unbroken PT- symmetric state as long as the eigenvalues of the mass matrix remain real,

$$M_{\pm}^2 = \frac{1}{2}(m_1^2 + m_2^2) \pm \sqrt{(m_1^2 - m_2^2)^2 - 4\mu^4}$$

- For $|m_1^2 - m_2^2| \geq 2\mu^2$: Unbroken PT symmetry.
- For $|m_1^2 - m_2^2| < 2\mu^2$: happens, Broken PT-symmetry phase as eigenvalues turn complex and PT $\psi_{\pm} \neq \pm\psi_{\pm}$.
  $\psi_{\pm}$: eigenfunctions of the mass matrix M^2 corresponding to eigenvalues $M_{\pm}^2$.
- We shall encounter similar non Hermitian mass matrix for gluons in our non Abelian model to be discussed.

CPT: Non-Hermitian Scalar Theory

- Note the eigenvalues $M^2_{\pm}$ do not change under $\mu^2 \to -\mu^2$, there still exist the charge conjugation symmetry under which the theory is CPT invariant in both PT broken and unbroken phases.
- The charge conjugation is defined as follows

$$\Phi \xrightarrow{\text{C}} C\Phi^*$$

  with C=P.
- The theory is CPT invariant in both PT broken and unbroken phases. In broken PT phase, $|m_1^2 - m_2^2| < 2\mu^2$ the theory violates CP also but preserves CT symmetry.
- Such charge conjugation symmetry exist in the non Abelian model also.

Non-Abelian Gauge Theory: Abelian Projection

- At low energy scale QCD can effectively expressed in terms of Abelian degree of freedom.
- If off diagonal gluons attain large mass, at IR below this mass scale off diagonal gluons become inactive and hence decouple. Dominant degrees of freedom are diagonal gluons as they remain massless.
- Thus SU(N) gauge theory reduces to N-1 copies of Abelian gauge theory. (Abelian projection)
- Abelian dominance is generally studied in a partial gauge fixing, known as Maximal Abelian gauge.

New Quadratic Gauge

- In a recent work Raval et al introduced an quadratic gauge $F^a[A^\mu(x)] = A^a_\mu A^{\mu a}$ for each a.
- This gauge is used to show the mass generation of off diagonal gluons and very useful to show the Abelian dominance.
- The effective Lagrangian

$$-\frac{1}{4}F^a_{\mu\nu}F_{a\mu\nu} - \frac{1}{2\xi}A^a_\mu A^{a\mu} - \bar{c}^a A^{\mu a}(D_\mu c)^a$$

.

- No ghost propagator, but ghost interacts with gluon. When the ghost condensates the Gluons acquire mass dynamically

$$(M^2)^{ab} = 2g \sum_{c=1}^{N^2-1} f^{abc} < \bar{c}^a c^c > .$$

- Diagonal gluons are massless due to anti-symmetric properties of f^{abc}. To obtain masses of gluons, we must diagonalize the matrix and find eigenvalues.

- Two different phases (i) the normal or deconfined phase and (ii) the ghost condensed phase showing the confinement.
- The Lagrangian in normal phase is given

$$-\frac{1}{4}F^a_{\mu\nu}F_{a\mu\nu} - \frac{1}{2\xi}A^a_\mu A^{a\mu} - \bar{c}^a A^{\mu a}(D_\mu c)^a$$

- We should note that the ghost Lagrangian does not have kinetic terms. They act like auxiliary fields in the normal phase, but play an important role in the IR regime.
- The ghost Lagrangian contains a term $gf^{abc}\bar{c}^a c^c A^{\mu a}A^b_\mu$. In this expression, ghost bilinears multiply the terms quadratic in gauge fields.
- Hence if the ghosts freeze they amount to a non-zero mass matrix for the gluons as follows

$$(M^2)^{ab}_{dyn} = 2g\sum_{c=1}^{N^2-1} f^{abc}\langle\overline{c^a}c^c\rangle. \tag{40}$$

Non-Hermitian Mass Matrix

- In an $SU(N)$ symmetric state, where all ghost-anti-ghost condensates are identical and ,

$$\begin{aligned} \langle \overline{c^1} c^1 \rangle = & \ldots = \langle \overline{c^1} c^{N^2-1} \rangle = \ldots = \langle \overline{c^{N^2-1}} c^1 \rangle = \ldots \\ = & \langle \overline{c^{N^2-1}} c^{N^2-1} \rangle \equiv K. \end{aligned} \quad (41)$$

- the mass matrix becomes

$$(M^2)^{ab}_{dyn} = 2gK \sum_{c=1}^{N^2-1} f^{abc} \quad (42)$$

which is an antisymmetric matrix i.e., non Hermitian,

$$(M^2)^{\dagger} \neq M^2$$

due to the antisymmetry of the structure constants.

QCD Phases in Quadratic gauge

- The mass matrix is unique, it has $N(N-1)$ non-zero eigenvalues only. $N(N-1)$ off-diagonal gluons obtain masses and the $N-1$ diagonal gluons remain massless.
- Due to antisymmetry, eigenvalues are purely imaginary, in conjugate pairs.
- The massive off-diagonal gluons are inferred as evidence of Abelian dominance, which is one of the signatures of quark confinement.
- Further, mass squared of the off-diagonal gluon is purely imaginary, hence the pole of the off-diagonal gluon propagator is on imaginary p^2 axis which is another important signature of color confinement.
- The mass for gluons generated through a given dynamical mechanism breaks the gauge symmetry as usual. There exist other mechanisms to provide mass to gluons in a gauge invariant manner. Thus, we see that the ghost condensation acts as the QCD vacuum.

Gluon Masses

- In the ghost condensed phase the Lagrangian can effectively be given as follows

$$\mathcal{L}_{GC} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} - \frac{1}{2\zeta}(A^a_\mu A^{\mu a})^2 + M_a^2 A^a_\mu A^{\mu a} \qquad (43)$$

- Mass for the diagonal gluons $M_a^2 = 0$, e.g, for $SU(3)$, $M_3^2 = M_8^2 = 0$.
- The off-diagonal gluons, $M_1^2 = +im_1^2, M_2^2 = -im_1^2,\ M_4^2 = +im_2^2, M_5^2 = -im_2^2,\ M_6^2 = +im_3^2, M_7^2 = -im_3^2$ (m_1^2, m_2^2, m_3^2 are positive real).
- The gluons 1 and 2 can be considered as conjugate of each other. The same is true for other pairs. Hence for $SU(3)$, the last term of the effective Lagrangian in Eq. (43) would be

$$\begin{aligned} M_a^2 A^a_\mu A^{\mu a} = \ & + \ im_1^2 A^1_\mu A^{\mu 1} - im_1^2 A^2_\mu A^{\mu 2} + im_2^2 A^4_\mu A^{\mu 4} - im_2^2 A^5_\mu A^{\mu 5} \\ & + \ im_3^2 A^6_\mu A^{\mu 6} - im_3^2 A^7_\mu A^{\mu 7} \end{aligned} \qquad (44)$$

Hermiticity Properties

- We first discuss the hermiticity property of two different phases. The effective Lagrangian in the normal phase is

$$\mathcal{L}_{eff} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} - \frac{1}{2\zeta}(A^a_\mu A^{\mu a})^2 - \overline{c^a}A^{\mu a}(D_\mu c)^a \quad (45)$$

- Now the hermiticity property of fields A^a_μ is well defined since they describe real degrees of freedom. Fields must be Hermitian in order to define the real degrees of freedom i.e.,

$$A^{a\dagger}_\mu = A^a_\mu \quad (46)$$

- The hermiticity of ghost fields remain unclear. As the operation of conjugation in principle transforms particle to its anti particle, the following is the natural choice of hermiticity property for ghosts. [Kugo & Ojima 1979]

- The natural choice of hermiticity property for ghosts

$$\begin{aligned} c^{a\dagger} &= \overline{c^a} \\ \overline{c^a}^{\dagger} &= c^a \end{aligned} \tag{47}$$

- Under these hermiticity properties of the fields

$$(f^{abc}\overline{c^a}c^c)^\dagger = f^{abc}\overline{c^c}c^a = f^{abc}\overline{c^a}c^c$$

(i.e., Hermitian) but the following ghost term is not Hermitian since

$$(\overline{c^a}\partial_\mu c^a)^\dagger = (\partial_\mu \overline{c^a})c^a \neq \overline{c^a}\partial_\mu c^a.$$

- Lagrangian in the normal phase is not Hermitian.
- The effective Lagrangian in the ghost condensed (confinement) phase is also not Hermitian as the mass term for gluons is purely imaginary as explained.

- The non hermiticity of the Lagrangian in this ghost condensed phase is free of the hermiticity convention for ghosts as they do not appear in this phase and thus the non hermiticity of the ghost condensed phase is profound.
- The Lagrangian in the ghost condensed phase obeys the extended hermiticity. i.e., when the following inner automorphisms is applied, hermiticity gets restored viz.
  $\mathfrak{T}\mathcal{L}^{\dagger}_{GC}\mathfrak{T}^{\dagger} = \mathcal{L}_{GC}$,

$$\begin{array}{llll} \mathfrak{T}L_1\mathfrak{T}^{\dagger} = L_2 & \mathfrak{T}L_4\mathfrak{T}^{\dagger} = L_5 & \mathfrak{T}L_6\mathfrak{T}^{\dagger} = L_7 & \mathfrak{T}L_3\mathfrak{T}^{\dagger} = L_8 \\ \mathfrak{T}L_2\mathfrak{T}^{\dagger} = L_1 & \mathfrak{T}L_5\mathfrak{T}^{\dagger} = L_4 & \mathfrak{T}L_7\mathfrak{T}^{\dagger} = L_6 & \mathfrak{T}L_8\mathfrak{T}^{\dagger} = L_3 \end{array} \quad (4$$

with the property

$$\mathfrak{T}^2 = \mathfrak{T}^{\dagger 2} = 1 \quad (49)$$

whereL_i refers to any individual Lagrangian term such as $-\frac{1}{4}F^i_{\mu\nu}F^{\mu\nu i}$, $-\frac{1}{2\zeta}(A^i_\mu A^{\mu i})^2$, $im^2 A^i_\mu A^{\mu i}$ appearing in Eq. (43).

Inner Automorphisms

- The inner automorphism amounts to exchanging group indices as $1 \leftrightarrow 2$, $4 \leftrightarrow 5$, $6 \leftrightarrow 7$, $3 \leftrightarrow 8$. In the adjoint representation it is given by

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- We have thus shown that both the normal and confined phases are non-Hermitian, later being profoundly.

Parity of fields

- For gluons, parity is given as

$$A_i^a(\mathrm{x}, t) \xrightarrow{\mathrm{P}} -A_i^a(-\mathrm{x}, t)$$
$$A_0^a(\mathrm{x}, t) \xrightarrow{\mathrm{P}} A_0^a(-\mathrm{x}, t).$$

  The rule for parity is same for all gluons as it is a linear operator.

- It is easy to see that Lagrangian in the normal phase is invariant under parity if we choose ghosts to be pseudo scalars,

$$c^a(\mathrm{x}, t) \xrightarrow{\mathrm{P}} -c^a(-\mathrm{x}, t)$$
$$\overline{c^a}(\mathrm{x}, t) \xrightarrow{\mathrm{P}} -\overline{c^a}(-\mathrm{x}, t).$$

- The ghosts being scalars under parity is equally acceptable for parity invariance of normal phase. However, we feel that the parity transformation above is more suitable as ghosts are anti-commuting scalars.

Time Reversal of Gluons

- The case of the time reversal is not straight forward unlike parity as the time reversal is an anti-linear operation. Since some of the generators of $SU(N)$ are purely imaginary, the time reversal property is not same for all gluons.
- For $SU(3)$ three generators namely, 2nd, 5th and 7th are purely imaginary. Time reversal for gluons is given by

$$A_i^p(\mathrm{x}, t) \xrightarrow{\mathrm{T}} -A_i^p(\mathrm{x}, -t)$$
$$A_0^p(\mathrm{x}, t) \xrightarrow{\mathrm{T}} A_0^p(\mathrm{x}, -t),$$

where index p is $1, 3, 4, 6, 8$ and,

-

$$A_i^q(\mathrm{x}, t) \xrightarrow{\mathrm{T}} A_i^q(\mathrm{x}, -t)$$
$$A_0^q(\mathrm{x}, t) \xrightarrow{\mathrm{T}} -A_0^q(\mathrm{x}, -t),$$

where index q is $2, 5, 7$

Time Reversal of Ghosts

- The field strength with any spacetime and group indices can utmost change up to overall negative sign i.e.,

$$F^a_{\mu\nu} \xrightarrow{\mathrm{T}} \pm F^a_{\mu\nu}.$$

- The action in the normal phase is invariant under time reversal given that the time reversal property for ghosts is defined in the following manner,

$$c^p(\mathrm{x}, t) \xrightarrow{\mathrm{T}} ic^p(\mathrm{x}, -t)$$
$$\overline{c^p}(\mathrm{x}, t) \xrightarrow{\mathrm{T}} i\overline{c^p}(\mathrm{x}, -t)$$

and,

$$c^q(\mathrm{x}, t) \xrightarrow{\mathrm{T}} c^q(\mathrm{x}, -t)$$
$$\overline{c^q}(\mathrm{x}, t) \xrightarrow{\mathrm{T}} \overline{c^q}(\mathrm{x}, -t)$$

- Anti-linearity makes two sets of ghosts transform in a completely different manner. Thus, the theory in normal phase is individually both parity and time reversal invariant. This PT symmetry breaks down spontaneously in the confined phase .

$$\mathcal{L}_{GC} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} - \frac{1}{2\zeta}(A^a_\mu A^{\mu a})^2 + M_a^2 A^a_\mu A^{\mu a}$$

- It is easy to check that parity is still a symmetry. However, the time reversal is broken due to pure complex nature of the mass term,

PT Phase transition

- $$\begin{aligned} M_a^2 A_\mu^a A^{\mu a} = & + im_1^2 A_\mu^1 A^{\mu 1} - im_1^2 A_\mu^2 A^{\mu 2} + im_2^2 A_\mu^4 A^{\mu 4} - im_2^2 A_\mu^5 A^{\mu 5} \\ & + im_3^2 A_\mu^6 A^{\mu 6} - im_3^2 A_\mu^7 A^{\mu 7} \quad \xrightarrow{\text{T}} \\ & - im_1^2 A_\mu^1 A^{\mu 1} + im_1^2 A_\mu^2 A^{\mu 2} - im_2^2 A_\mu^4 A^{\mu 4} + im_2^2 A_\mu^5 A^{\mu 5} \\ & - im_3^2 A_\mu^6 A^{\mu 6} + im_3^2 A_\mu^7 A^{\mu 7} \\ & = -M_a^2 A_\mu^a A^{\mu a} \end{aligned} \tag{50}$$
- Also PT$\psi \neq \pm\psi$, where ψs are eigenfunctions of the mass matrix. The first two terms of $\mathcal{L}_{GC}$ remain unaffected by the time-reversal. Thus, PT symmetry is violated in this phase.
- We can see that the anti symmetric nature of structure constant appearing in the mass matrix has led to this breaking. Important point again here is to note that the PT symmetry violation in the confined phase is profound as it is independent of the convention for ghosts.

PT phase transition

- The transition from the normal phase to the confinement phase with $SU(N)$ symmetric ghost condensates thus is identified as PT phase transition from unbroken to broken.
- Association between color confinement and spontaneous PT breaking is model and mechanism independent even though in this model the link is through ghost condensation.
- Since one prime signature of quark confinement, the pole of the propagator on purely imaginary p^2 axis, inevitably breaks PT symmetry. The model gives valuable insight into a process through which the link can take place.

Order Parameter

- There is a crucial difference between the non Abelian model and the toy model of complex scalars. Complex scalar theory has the parameter $\eta \equiv \frac{2\mu^2}{|m_1^2 - m_2^2|}$ whose value separates two phases of the PT symmetry in the theory.
- There is no such single order parameter in the non Abelian theory which governs the phase transition. Different ghost bilinears $\overline{c^a}c^c$ (a and c runs over 1 to $N^2 - 1$ independently) gradually condensing to the $SU(N)$ symmetric vacuum state give rise to the PT phase transition in this non Abelian model.

CPT symmetry

- We show that in the setup of quantum field theory in which we are working the inner automorphism provides the representation of this C-symmetry. So far, no explicit representation of the C-symmetry is known in the framework of gauge theories. This symmetry in quantum mechanics must satisfy the following three conditions

$$[H, C]\psi = 0, \ [PT, C]\psi = 0, \ C^2 = \mathbf{1}$$

- The inner automorphism satisfies QFT analogue of the above conditions .(1) The inner automorphism exchanges group indices i.e., $1 \leftrightarrow 2$, $4 \leftrightarrow 5$, $6 \leftrightarrow 7$, $3 \leftrightarrow 8$ and the Lagrangian of the initial unbroken PT theory in the normal phase contains sum over group index a. Hence, the Lagrangian and therefore Hamiltonian in this phase remain invariant under the inner automorphism. Thus, QFT analogue of the first of conditions is obeyed.

CPT symmetry

- (2) PT is a space-time symmetry and the inner automorphism is the operation in the group space. Therefore, it is easy to check that changing the order of inner automorphism and PT operations on Lagrangians of both the phases does not alter the final result. In other words, they commute. This proves the QFT analogue of the second condition.
- (3) The third of condition has already been shown. Therefore, we see that the inner automorphism forms an explicit representation of the C-symmetry, which in adjoint representation is given by the matrix.
- It is clear that the theory in both the phases is invariant under CPT. In the broken PT phase, the theory also violates CP symmetry but preserves the CT, in complete analogy with the scalar model

Conclusion/Summary

- PT Phase transition in 1-d QES system was realised very early.
- PT phase transition is higher dimension has many interesting features. In the case of 2-d anisotropic oscillator, Phase transition occurs as long as the oscillator is anisotropic. Indicates possible link of PT phase transition and symmetry of the original Harmitian Hamiltonian.
- Due to the discovery of graphene, relativistic QM on the plane has become more interesting. We demonstrate PT phase transition in Dirac Oscillator in the background magnetic field with imaginary Spin orbit coupling. We have unbroken PT phase when the strength of the coupling is less than certain critical value and alternatively when strength of the magnetic fields is greater than certain critical value.

Conclusion/Summary

- Our result shows the existence of mass gap in graphene consistent with the experimental results in the unbroken phase.
- The first and novel example of non-Hermitian gauge theory exhibiting PT phase transition is presented.
- Natural hermiticity property of ghost fields is adopted.
- Transition between two QCD phases is identified as PT phase transition. Ghost condensation plays the role of order parameter in this.
- C-symmetry and its explicit representation is identified. Hence, the present theory is consistent non-Hermitian gauge theory.

Thanks for your Attention

ICTS, Bangaluru, when we had our 18 th meeting in PHHQP, recently wrote me that our meeting was amongst the most successful and impactful ICTS programs in the year 2018-19.
In view of that they have requested me to organise a short follow up meeting (in the Virtual mode) with recent developments in this field. We are thinking of organising a 4 day meeting during last week of February or 1st week of March, 2021.
Please let me (bhabani.mandal@gmail.com) know if you are interested to give a talk in that meeting.