Unmasking classical $PT$ symmetry

Work in progress with Daniel Hook

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Thank you Andreas and Francisco for organizing this lecture series, but I wish I were giving this talk live in New Mexico…

This is the first time I have ever given a physics talk sitting down!

These are truly depressing times…
More than any time in history mankind stands at a crossroads. One path leads to despair and utter hopelessness, the other to total extinction. Let us pray we have the wisdom to choose correctly.

-- Woody Allen
The model: One-parameter family of PT-symmetric Hamiltonians obtained by complex deformation of the harmonic oscillator

\[ H = p^2 + x^2(ix)^\varepsilon \] (\( \varepsilon \) real)

Special cases:
- Cubic: \( H = p^2 + ix^3 \)
- Quartic: \( H = p^2 - x^4 \)
- Sextic: \( H = p^2 + x^6 \)

Look! \( H \) is not Hermitian but its eigenvalues are all real if \( \varepsilon > 0 \!\!\)!

P. Dorey, C. Dunning, and R. Tateo

P. Dorey, C. Dunning, and R. Tateo
Values of $\varepsilon$ at which a sudden quantum change occurs:

(1) *PT phase transition* at $\varepsilon = 0$, where complex energies abruptly change from complex to real!

(2) Unique special value $\varepsilon = 2$ at which there is a *closed-form* similarity mapping from the non-Hermitian quartic Hamiltonian $H$ to an isospectral *local* Hermitian quartic Hamiltonian $h$: $h = S^{-1} HS$

Hook and I decided to *try* to explore these abrupt cataclysmic quantum changes. We were going to call our paper: “*Try simple cataclysms*”

But…
To our surprise, if we rearrange the letters of

“Try simple cataclysms” we get:

“Classical $PT$ symmetry”

So, we decided that we should see what happens at the classical level.
We unmasked a rich variety of previously overlooked classical phenomena...

(1) Infinite sequences of *separatrix* trajectories

(2) Infinite sequences of critical initial values associated with limiting classical orbits

(3) Regions of broken *PT*-symmetric trajectories

(4) Topological transition at $\varepsilon = 2$, and perhaps at other even-integer values of $\varepsilon$ as well
Known classical topological transition:

*Open* orbits for $\varepsilon < 0$; *closed* orbits for $\varepsilon > 0$

Initial condition:

$x(0) = -0.2i$

**Bohr-Sommerfeld**
Quantization of a complex atom

A complex integral along a *closed* path:

$$\oint dx \: p = (n + \frac{1}{2})$$
This talk presents a new study of the classical particle trajectories for the Hamiltonian

\[ H = p^2 + x^2(ix)^\varepsilon \]

in the region of unbroken PT symmetry \( \varepsilon > 0 \).

Hamilton’s equations of motion:

\[
\frac{d}{dt}x = \frac{\partial}{\partial p} H = 2p,
\]

\[
\frac{d}{dt}p = -\frac{\partial}{\partial x} H = -(2 + \varepsilon)x(ix)^\varepsilon.
\]

Derive Newton’s equation of motion by eliminating \( \dot{p} \):

\[ \ddot{x} = -2(2 + \varepsilon)x(ix)^\varepsilon. \]
These differential equations are **complex**. So they give particle trajectories in the complex-$x$ plane. Simplest case is $\varepsilon = 0$ (harmonic oscillator): $H = p^2 + x^2$

Energy $E = 1$

All but one of the orbits are space-filling nested ellipses; *not* Keplerian because period of all orbits is $\pi$.

Critical *terminating* trajectory connecting the *turning points*

These trajectories are *closed*. Remember: Trajectories transition at $\varepsilon = 0$ from *open* in the *PT* broken region ($\varepsilon < 0$) to *closed* in the *PT* unbroken region ($\varepsilon > 0$).
For \( \varepsilon > 0 \), turning points move down into complex plane:

\[
x_{\text{right}} = \exp\left(-\frac{i\pi \varepsilon}{2\varepsilon + 4}\right) \quad \text{and} \quad x_{\text{left}} = \exp\left(-i\pi + \frac{i\pi \varepsilon}{2\varepsilon + 4}\right)
\]

Branch cut on the positive-imaginary axis:

Trajectories in Region \( R_0 \)
for \( \varepsilon = 0.3 \)

Last trajectory that does not cross branch cut; this curve bounds Region \( R_0 \)

Terminating orbit begins at \( s_0 = -0.309636i \)

Critical point \( x_0 = -0.6438554i \)

These nested trajectories fill Region \( R_0 \)
(1) Orbital period:

\[ T = \frac{1}{2} \int_C \frac{dx}{\sqrt{1 - x^2(ix)^\varepsilon}}. \]

For all orbits inside Region \( R_0 \):

\[ T = 2\sqrt{\pi} \cos\left(\frac{\pi\varepsilon}{4+2\varepsilon}\right) \Gamma\left(\frac{3+\varepsilon}{2+\varepsilon}\right) / \Gamma\left(\frac{4+\varepsilon}{4+2\varepsilon}\right). \]

(2) Slope of terminating orbit at (right) turning point:

\[ \ddot{x} = -2(2 + \varepsilon)x(ix)^\varepsilon \]

\[ \ddot{x} = -(4 + 2\varepsilon) \exp\left(i\frac{\pi\varepsilon}{4+2\varepsilon}\right). \]

So, terminating curve at right turning point slopes upward at angle \( \frac{\pi\varepsilon}{4+2\varepsilon} \).
\[ p^2 + x^2(ix)^\varepsilon = 1 \] for \( E = 1 \)

At \( p = 0, x^2(ix)^\varepsilon = 1 \) (turning points lie on unit circle)

First pair of turning points \( t_0 \):

\[
 x_{\text{right}} = \exp \left( -\frac{i\pi\varepsilon}{2\varepsilon+4} \right) \quad \text{and} \quad x_{\text{left}} = \exp \left( -i\pi + \frac{i\pi\varepsilon}{2\varepsilon+4} \right)
\]

Turning points \textit{pull} on trajectories in the complex plane

Saddle points \textit{pull and push} on trajectories in the (real) phase plane

**Generic turning point**

(1 \textit{complex} degree of freedom)

**Generic saddle point**

(2 \textit{real} degrees of freedom)
What happens to lower edge of Region $R_0$ as $\epsilon$ increases?

Critical point $x_0$ moves down the imaginary axis:

Transition at $\epsilon = 2$
Initial point $s_0$ of *terminating* trajectory also moves downward but only by a finite amount:
\[ \varepsilon = 0.3 \]

Trajectory just outside Region \( R_0 \)

Critical point \( x_0 = -0.6438554 i \)

Critical point \( x_1 = -6.309902 i \)

\( t_0 = \pm 0.9791 - 0.2035 i \)

\( t_1 = \pm 0.8170 + 0.5767 i \)

Purple on sheet 0, red on sheet 1, blue on sheet -1
Terminating trajectory inside Region $R_1$

$\epsilon = 0.3$

$s_1 = -2.1248i$
Note: Every closed trajectory encloses exactly 2 turning points; trajectories in Region $R_1$ do not enclose any points in Region $R_0$

Region $R_1$ does not enclose Region $R_0$
3D view of a trajectory in Region $R_1$

Purple on sheet 0, red on sheet 1, blue on sheet -1
Summary of properties

(1) Infinite sequence of $PT$-symmetric pairs of turning points $t_0, t_1, t_2, \ldots$ going clockwise and anticlockwise around the unit circle.

(2) Infinite sequence of critical initial conditions $x_0, x_1, x_2, \ldots$ on the negative-imaginary axis giving rise to separatrix curves. For $\varepsilon = 0.3$: $x_0 = -0.6438554 \, i$, $x_1 = -6.309902 \, i$, $x_2 = -729.348 \, i$, \ldots

(3) Separatrix curves separate $\text{Regions } R_0, \text{Regions } R_1, \text{Regions } R_2, \ldots$; Region $R_n$ bounded above and below on negative-imaginary axis by $x_{n-1}$ and $x_n$.

(4) Inside Region $R_n$ is a terminating curve that begins at special point $s_n$ on negative-imaginary axis. This curve joins turning points $t_n$. For $\varepsilon = 0.3$: $s_0 = -0.309636 \, i$, $s_1 = -2.12477 \, i$, $s_2 = -44.9112 \, i$, \ldots

(5) Regions $\{R_n\}$ form a disjoint cover of the entire Riemann surface.

(6) For irrational $\varepsilon < 1$, infinite towers of critical points $\{x_n\}$ and special points $\{s_n\}$. 
Example of an infinite tower

Trump Tower...
Treat critical points $x_n$ as **eigenvalues**; separatrix curves that emanate from $x_n$ as **eigenfunctions**

See earlier work with Andreas Fring, Javad Komijani, and Qinghai Wang

Example: $y'(t) = \cos[\pi ty(t)], \quad y(0) = a$

\[ y(0) = a \in \{1.6026, 2.3884, 2.9767, 3.4675, 3.8975, 4.2847, \ldots\} \]  \hspace{1cm} (eigenvalues)

\[ a_n \sim 2^{5/6} \sqrt{n} \quad (n \to \infty) \]
Another example: First Painlevé transcendent

\[ y''(t) = 6[y(t)]^2 + t, \quad y(0) = c, \: y'(0) = b \]

\[ c = 0, \quad b_n \sim Bn^{3/5} \quad (n \to \infty) \]

\[ B = 2 \left[ \sqrt{3\pi} \Gamma \left( \frac{11}{6} \right) / \Gamma \left( \frac{1}{3} \right) \right]^{3/5} \]

\[ b = 0, \quad c_n \sim Cn^{2/5} \quad (n \to \infty), \]

\[ C = - \left[ \sqrt{3\pi} \Gamma \left( \frac{11}{6} \right) / \Gamma \left( \frac{1}{3} \right) \right]^{2/5} \]

These results obtained from a WKB analysis of the PT-symmetric Hamiltonian \( H = p^2 + ix^3 \) (\( \epsilon = 1 \)). [Second Painlevé transcendent associated with \( H = p^2 - x^4 \) (\( \epsilon = 2 \)).]

But, surprise! The critical values \( x_n \) and \( s_n \) appear to grow \textit{exponentially} and not algebraically with \( n \).
This is the picture for \textit{irrational} \( \varepsilon \), where \( 0 < \varepsilon < 1 \). If \( \varepsilon = m/n \) is \textit{rational}, classical equation of motion has two symmetries, \textit{PT} symmetry and \( 2n\pi \) rotation symmetry because of \( x^\varepsilon \) term in Hamiltonian & equation of motion

Note: Even if a differential equation is defined on an \( n \)-sheeted Riemann surface, the solution may not be. Example: Solution to \( y'(x) = \frac{1}{x} \) is \( y(x) = \log(x) + c \)

But if \( \varepsilon < 1 \), \textit{the solution does appear to live on the same finite Riemann surface as the equation of motion!}

Some examples…
\[ \epsilon = \frac{1}{2} \]

**Five** turning points on a two-sheeted Riemann Surface:

- \( t_0 : \frac{-11}{10} \pi \) and \( \frac{-9}{10} \pi \) on sheet 0
- \( t_1 : \frac{7}{10} \pi \) and \( \frac{-17}{10} \pi \) on sheet 1 and -1
- \( t_2 : \frac{3}{2} \pi \) and \( \frac{-5}{2} \pi \) on sheet 1 and -1 (!)

Three terminating trajectories in three regions \( R_1, R_1, R_2 \), and the third **one-ended** trajectory terminates at \(-i \infty\)
Typical trajectory in each of the three regions for $\epsilon = 1/2$

Blue trajectory in Region $R_2$ winds on the two-sheeted Riemann surface. Set of blue trajectories is bounded by the red one-ended trajectory.
For $\varepsilon = 1/n$ there are:

$n$ sheets in the Riemann surface

$n + 3$ turning points (one unpaired if $n$ is even)

$n + 1$ regions
Special case $\varepsilon = 1 \ (n = 1)$
\[ \epsilon = \frac{2}{3} \]

8 turning points
3 sheets
4 regions

3-sheeted unit cell
Orbits with broken \textit{PT}

\[ \varepsilon = \frac{4}{3} \]
When $\varepsilon > 2$ and *rational* there are "bands" and "gaps" – sort of…

$\varepsilon = 7/3$

Terminating $PT$-symmetric path inside Region $R_0$
\( \varepsilon = \frac{7}{3} \)

Initial value

\( x(0) = -0.208i \)  
(\textit{below origin!})

Closed path very close to boundary of \textit{Region} \( R_0 \)
$\varepsilon = 7/3$

Terminating $PT$-symmetric path crossing negative-imaginary axis just above Region $R_0$
\( \varepsilon = \frac{7}{3} \)

**PT**-symmetric closed path containing previous terminating path close to but **above** upper edge of *Region \( R_0 \)* with \( x(0) = -0.172025i \). Between \(-0.172025i\) and \(-0.208i\) there is a *band* of thickness approximately \(0.035i\)
Blow-up of previous two paths – showing *band* between upper edge of *Region $R_0$* and another region!

$$\epsilon = 7/3$$
\( \epsilon = 7/3 \)

Winding path beginning in **band**. We have “gaps” (particles in closed periodic orbits) and “bands” (particles winding endlessly through a *crystal* lattice consisting of discrete sheets in the Riemann surface)
Conjecture: There is a transition at each new even-integer value of $\varepsilon$ at which there is one new band on the negative-imaginary axis, inside of which is a new family of trajectories that wind from sheet to sheet in the Riemann surface “crystal”
Thanks for listening!

I am happy to answer questions...