

CLASSICAL AND QUANTUM DYNAMICS OF  
HIGHER-DERIVATIVE SYSTEMS

OR

LIVING WITH GHOSTS

PHHQP seminar, July 23, 2020

based on

A.S., Int. J. Mod. Phys. A32 (2017) no.33,  
1730025 [arXiv:1710.11538]

## MOTIVATION:

*... To myself, I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary whilst the great ocean of truth lay all undiscovered before me.*

Isaac Newton

- The ocean is now charted up to  $E \lesssim 10^3$  GeV,  $l \gtrsim 10^{-17}$  cm.
- But it extends up to  $M_P \approx 10^{19}$  GeV. We have now explored its  $10^{-16}$ -th part.

## PROBLEMS IN QUANTUM (AND CLASSICAL) GRAVITY:

- Nonrenormalizability
- Non-causality. Closed time loops. Paradoxes.

## TOE = strings?

- No fundamental quantum string theory
- No phenomenological successes.

An alternative (dream) solution: [A.S., 2005]

*Our Universe as a soap film in a flat higher dimensional bulk. The TOE is a field theory in this bulk. Gravity etc is an effective theory living on the film, like*

$$H_{\text{soap}} = \sigma \mathcal{A} = \sigma \int d^2x \sqrt{g}$$

## TRY

$$S = -\frac{1}{2h^2} \int \text{Tr}\{F_{MN}F_{MN}\} d^6x,$$

in  $D = 6$ ,  $M, N = 0, 1, 2, 3, 4, 5$ .

- Dimensionful coupling constant, nonrenormalizable

## A SECOND TRY

$$\mathcal{L}^{D=6} = \alpha \text{Tr}\{F_{\mu\nu} \square F_{\mu\nu}\} + \beta \text{Tr}\{F_{\mu\nu} F_{\nu\alpha} F_{\alpha\mu}\}$$

- $\alpha, \beta$  are dimensionless, renormalizability
- Includes higher derivatives

## But GHOSTS appear

- In a ghost system the Hamiltonian has no ground state. No vacuum in field theory. It is inherent for all higher-derivative theories.

## OSTROGRADSKY HAMILTONIAN

M. Ostrogradsky [1801-1862] is known by

- Ostrogradsky theorem from vector analysis
- Ostrogradsky method for calculating  $\int \frac{P(x)}{Q(x)} dx$ .
- Ostrogradsky Hamiltonian

In the paper

[M. Ostrogradsky, *Mémoire sur les équations différentielles relatives au problème des isopérimètres*, Mem. Ac. St. Petersburg **VI 4** (1850) 385.]

he reinvented the Hamiltonian formalism and applied it to higher-derivative theories.

- Consider  $L(x, \dot{x}, \ddot{x})$ .
- Equation of motion:

$$\frac{d^2}{dt^2} \left( \frac{\partial L}{\partial \ddot{x}} \right) - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) + \frac{\partial L}{\partial x} = 0.$$

- Conserved energy:

$$E = \ddot{x} \frac{\partial L}{\partial \ddot{x}} + \dot{x} \left( \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{x}} \right) - L.$$

- Treat  $v = \dot{x}$  as an independent variable and define

$$p_v = \frac{\partial L}{\partial \dot{v}} = \frac{\partial L}{\partial \ddot{x}},$$

$$p_x = \frac{\partial L}{\partial \dot{x}} - \dot{p}_v,$$

- Canonical Hamiltonian:

$$H(p_v, p_x; v, x) = p_v \dot{v} + p_x \dot{x} - L =$$

$$p_v a(p_v, x, v) + p_x v - L[a(p_v, x, v), x, v],$$

where  $a(p_v, x, v)$  is the solution of the equation  $\partial L(x, v, a)/\partial a = p_v$ .

- Linear term  $p_x v \longrightarrow$

**Theorem 1:** [Woodard, 2015] *The classical energy of a nondegenerate higher-derivative system can acquire an arbitrary positive or negative value.*

also

**Theorem 2:** [Raidal + Veermae, 2017] *The spectrum of a Hamiltonian of a higher-derivative system is not bounded neither from below, nor from above.*

- This phenomenon is sometimes called “Ostrogradsky instability”, but:
  - It is not an instability,
  - was noticed first not by Ostrogradsky.

**Arnold’s principle:** *If a notion bears a personal name, then this name is not the name of the discoverer.*

(self-referential).

## PAIS-UHLENBECK OSCILLATOR [1950]

Consider

$$L = \frac{1}{2} [\ddot{x}^2 - (\omega_1^2 + \omega_2^2)\dot{x}^2 + \omega_1^2\omega_2^2x^2]$$

- Ostrogradsky Hamiltonian:

$$H = p_x v + \frac{p_v^2}{2} + \frac{(\omega_1^2 + \omega_2^2)v^2}{2} - \frac{\omega_1^2\omega_2^2x^2}{2}$$

- Canonical transformation [Mannheim + Davidson, 2005]:

$$\begin{aligned} X_1 &= \frac{1}{\omega_1} \frac{\hat{p}_x + \omega_1^2 v}{\sqrt{\omega_1^2 - \omega_2^2}}, & \hat{P}_1 &\equiv -i \frac{\partial}{\partial X_1} = \omega_1 \frac{\hat{p}_v + \omega_2^2 x}{\sqrt{\omega_1^2 - \omega_2^2}}, \\ X_2 &= \frac{\hat{p}_v + \omega_1^2 x}{\sqrt{\omega_1^2 - \omega_2^2}}, & \hat{P}_2 &\equiv -i \frac{\partial}{\partial X_2} = \frac{\hat{p}_x + \omega_2^2 v}{\sqrt{\omega_1^2 - \omega_2^2}}. \end{aligned}$$

( $\omega_1 > \omega_2$  was assumed).

- In these variables,

$$H = \frac{\hat{P}_1^2 + \omega_1^2 X_1^2}{2} - \frac{\hat{P}_2^2 + \omega_2^2 X_2^2}{2}.$$

[Pauli, 1943]

The spectrum is

$$E_{nm} = \left(n + \frac{1}{2}\right) \omega_1 - \left(m + \frac{1}{2}\right) \omega_2$$



with positive integer  $n, m$ .

- All states are normalizable (“pure point”). Infinite degeneracy if  $\omega_1/\omega_2 \neq 1$  is rational. Everywhere dense if  $\omega_1/\omega_2$  is irrational.

**Degenerate case**  $\omega_1 = \omega_2$

- The canonical transformation is singular.
- Use

$$\hat{H} = v\hat{p}_x + \frac{\hat{p}_v^2}{2} + \omega^2 v^2 - \frac{\omega^4 x^2}{2}.$$

- Continuum spectrum  
[Pais + Uhlenbeck, 1950; Bolonek + Kosinski, 2006]

$$\Psi_{kl}(x, v; t) = e^{-i\omega^2 xv} \exp \left\{ \frac{i}{4\omega^2} \frac{\partial^2}{\partial x \partial v} \right\} \left[ J_l(k\sqrt{v^2 + \omega^2 x^2}) \left( \frac{\omega x - iv}{\omega x + iv} \right)^{1/2} \right] e^{-it(l\omega + k^2/4)}.$$

- The energy  $E = l\omega + k^2/4$  may be positive, negative, or zero.

**UNUSUAL BUT NOT SICK!**

## UNITARITY CONFUSION: WHY "GHOSTS" ?

- Negative kinetic energy as for Faddeev-Popov ghosts.

Consider

$$\hat{H} = \omega_1 a_1^\dagger a_1 - \omega_2 a_2 a_2^\dagger.$$

- Ordinary "vacuum"  $|\Phi\rangle$  with  $a_1|\Phi\rangle = a_2|\Phi\rangle = 0$  is in the middle of the spectrum.
- Introduce the state  $|\tilde{\Phi}\rangle$  satisfying

$$a_1|\tilde{\Phi}\rangle = a_2^\dagger|\tilde{\Phi}\rangle = 0$$

and consider the tower of states

$$|n\rangle = \frac{a_2^n}{n!}|\tilde{\Phi}\rangle$$

Then  $\hat{H}|\tilde{\Phi}\rangle = 0$ ,

$$\hat{H}|1\rangle = \hat{H} a_2|\tilde{\Phi}\rangle = -\omega_2 a_2 a_2^\dagger a_2|\tilde{\Phi}\rangle = \omega_2 a_2|\tilde{\Phi}\rangle = \omega_2|1\rangle$$

and  $\hat{H}|n\rangle = n\omega_2|n\rangle$ .

- The spectrum is positive definite. One can rename  $a_2 \rightarrow b_2^\dagger$ ,  $a_2^\dagger \rightarrow b_2$ .

## THE PRICE

- $[b_2, b_2^\dagger] = -1$  and hence  $|1\rangle = b_2^\dagger|\tilde{\Phi}\rangle$  has a **negative norm**:

$$\langle 1|1\rangle = -\langle\tilde{\Phi}|b_2b_2^\dagger|\tilde{\Phi}\rangle = -\langle\tilde{\Phi}|\tilde{\Phi}\rangle = -1.$$

- 

$$|\tilde{\Phi}\rangle = \exp\left\{-\frac{\omega_1}{2}X_1^2\right\}\exp\left\{\frac{\omega_2}{2}X_2^2\right\}$$

and the whole tower  $|n\rangle$  are not in  $\mathcal{L}_2$ .

**IT IS BETTER NOT TO THINK IN THESE TERMS!**

## BENDER AND MANNHEIM PROPOSAL [2008]

Consider

$$H = \frac{\hat{P}_1^2 + \omega_1^2 X_1^2}{2} - \frac{\hat{P}_2^2 + \omega_2^2 X_2^2}{2}. \quad (1)$$

and assume  $X_1$  to be real and  $X_2$  purely imaginary.

Then the **normalizable** wave functions involve a factor

$$\exp \left\{ -\frac{\omega_1}{2} X_1^2 \right\} \exp \left\{ \frac{\omega_2}{2} X_2^2 \right\}.$$

and the spectrum is positive definite.

- This is a **different** spectral problem.
- No real need to do this: the free Hamiltonian (1) is by itself benign and Hermitian. The evolution operator is unitary!

## PATH INTEGRAL CONFUSION

- The Minkowski Lagrangian path integral

$$\sim \int \prod_t dx(t) \exp \left\{ i \int dt L(\ddot{x}, \dot{x}, x) \right\} \quad (2)$$

- The corresponding Hamiltonian path integral

$$\sim \int \prod_t dx(t) dv(t) dp_x(t) dp_v(t) \exp \left\{ i \int dt [p_v \dot{v} + p_x \dot{x} - H(p_v, p_x; v, x)] \right\}. \quad (3)$$

Substitute here the Ostrogradsky Hamiltonian and integrate over  $\prod_t dp_x(t)$ . We obtain the factor

$$\prod_t \delta[v(t) - \dot{x}(t)].$$

Integrating further over  $\prod_t dv(t) dp_v(t)$ , we derive (2).

- The Euclidean rotation  $t \rightarrow -i\tau$  in the Hamiltonian integral (3) is **impossible**. The integral

$$\prod_\tau \int_{-\infty}^{\infty} dp_x(\tau) \exp \left\{ \int d\tau p_x(\tau) \left[ i \frac{dx(\tau)}{d\tau} - v(\tau) \right] \right\}$$

diverges.

- The Euclidean evolution operator and partition function

$$Z = \sum_n e^{-\beta E_n}$$

are not defined.

- Euclidean rotation  $t \rightarrow -i\tau$  in the Lagrangian integral (2) is possible, the integral may converge, but its analytic continuation into Minkowski space does not give a unitary evolution. [Hawking + Hertog, 2002; Andrzejewski + Gonera + Maslanka, 2009]

### CONCLUSION:

*Euclidean path integrals (in contrast to Minkowski ones) are not defined for higher-derivative systems.*

## INCLUDING INTERACTIONS

Consider the Lagrangian [A.S., 2005]

$$L = \frac{1}{2} [\dot{x}^2 - 2\omega^2 \dot{x}^2 + \omega^4 x^2] - \frac{1}{4} \alpha x^4.$$

Equation of motion:

$$\left( \frac{d^2}{dt^2} + \omega^2 \right)^2 x - \alpha x^3 = 0.$$

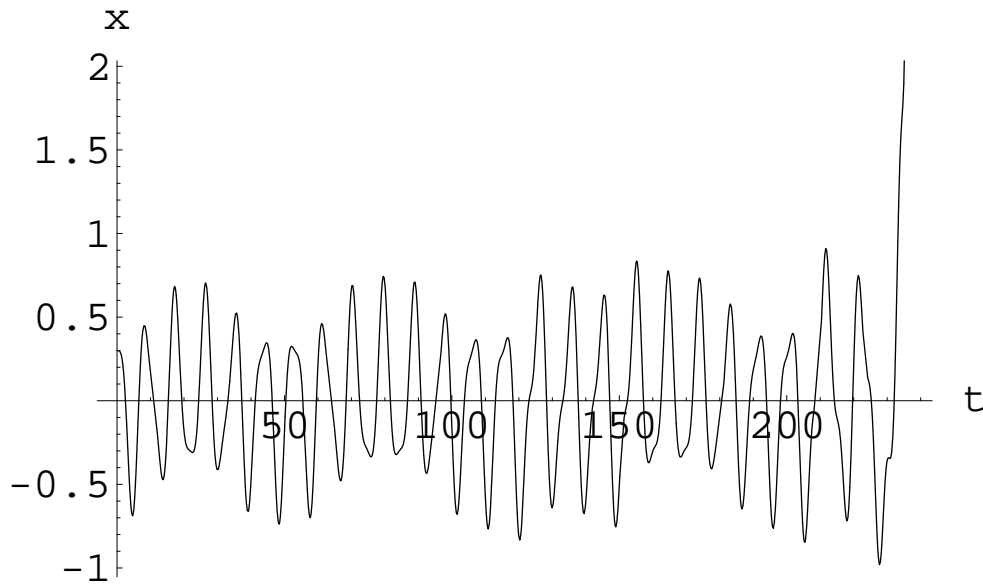
fixed point:

$$x(0) = \dot{x}(0) = \ddot{x}(0) = x^{(3)}(0) = 0.$$

- **Stable trajectories** for small deviations (the island of stability).
- **Collapse** (the system runs to the infinity at finite time) for large enough deviations.



## ON THE SHORE OF THE STABILITY ISLAND



A similar stability island for another HD system in

[S.N. Carrol, M. Hoffman, and M. Trodden,  
PR **D68** (2003) 023509]

## AN EXAMPLE OF COLLAPSE: FALLING TO THE CENTER

Consider

$$V(r) = -\frac{\kappa}{r^2}$$

with  $m\kappa > 1/8$ .

- Spectrum is not bounded from below.
- Schrödinger problem is not well defined.

If one smoothes the singularity,

$$V(r) = -\frac{\kappa}{r^2}, \quad r > a,$$
$$V(r) = -\frac{\kappa}{a^2}, \quad r \leq a,$$

the spectrum is bounded, but depends on  $a$ .

- Violation of unitarity (probability “leaks” into the singularity).

### AN OBSERVATION:

- If quantum theory is sick, so is its classical counterpart. If classical theory is benign, so is its quantum counterpart.

# INTERACTING SYSTEMS WITH BENIGN GHOSTS

[D. Robert + A.S., 2006]

$$S = \int dt d\bar{\theta} d\theta \left[ \frac{i}{2} \bar{\mathcal{D}}\Phi \frac{d}{dt} \mathcal{D}\Phi + V(\Phi) \right],$$

with the real (0+1)-dimensional superfield

$$\Phi = \phi + \theta\bar{\psi} + \psi\bar{\theta} + D\theta\bar{\theta}$$

- An **extra** time derivative.

## The Hamiltonian

$$H = pP - DV'(\phi) + \text{fermion term}$$

is not positive definite.

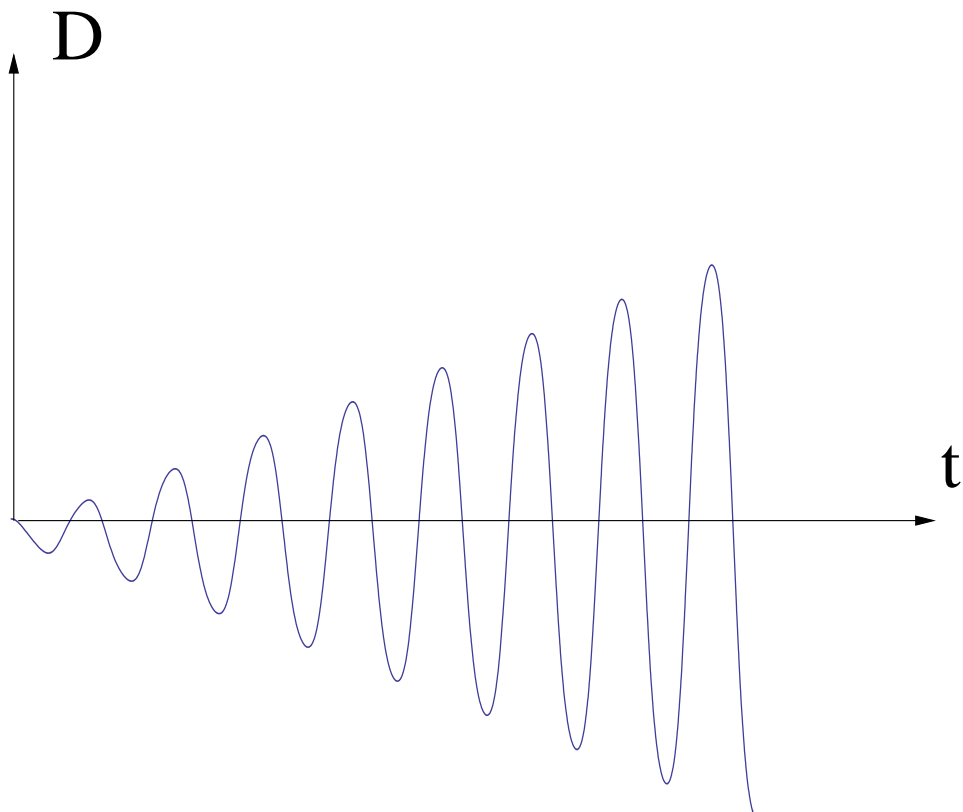
- 4-dimensional phase space  $(p, \phi), (P, D)$ .
- Two integrals of motion:  $H$  and

$$N = \frac{P^2}{2} + V(\phi).$$

- Exactly solvable.
- Take

$$V = \frac{\omega^2 \phi^2}{2} + \frac{\lambda \phi^4}{4}$$

- The solutions to the classical equations of motion are expressed via elliptic functions.

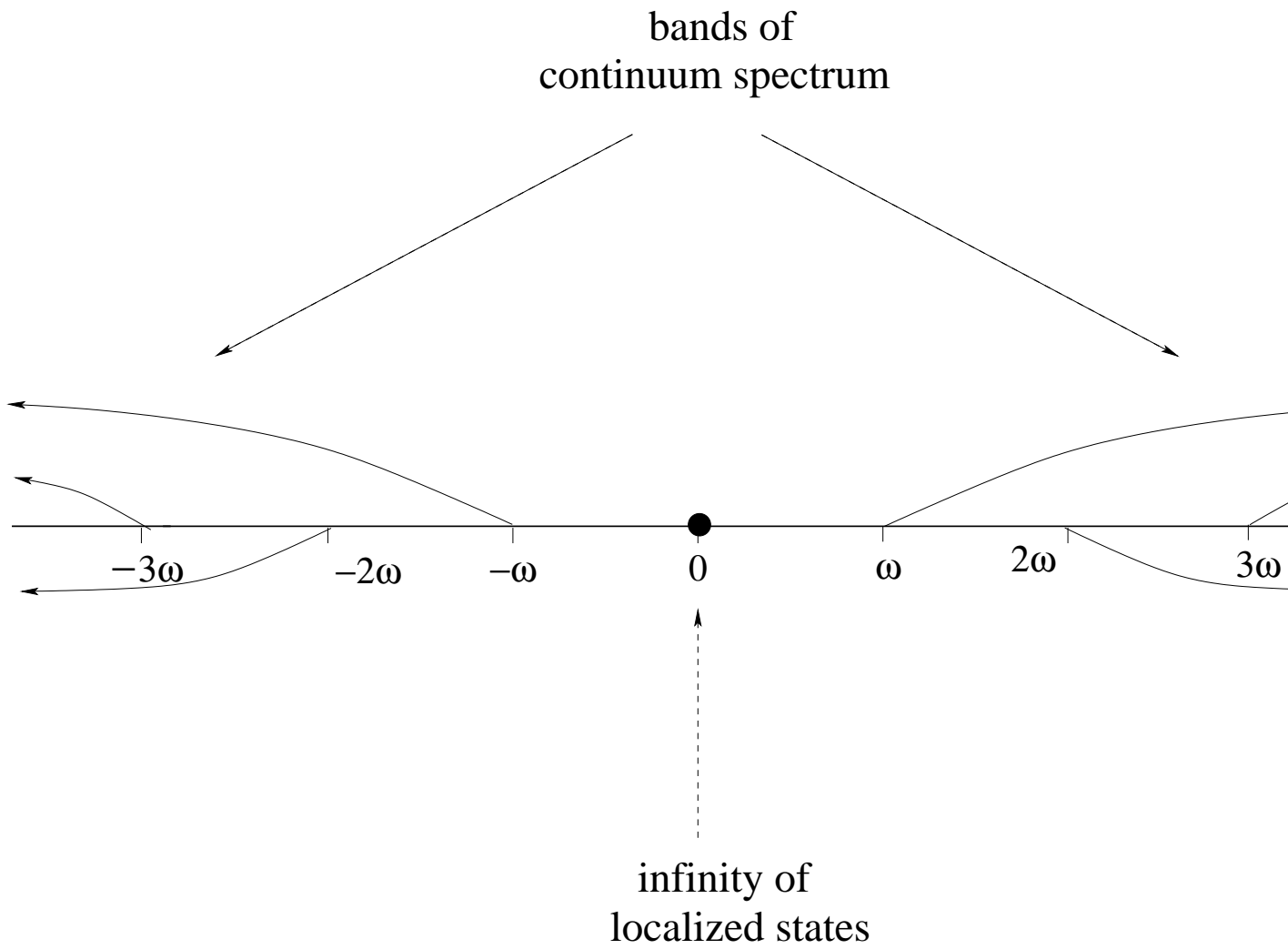


- Linear growth for  $D(t)$ ;  $\phi(t)$  is bounded. No collapse.

- Other benign ghost systems:

[Pavšič, 2013; Ilhan+Kovner, 2013]

QUANTUM PROBLEM  
is also exactly solvable.



Spectrum of the Hamiltonian  $\hat{H} = \hat{p}\hat{P} + DV'(\phi)$ .

## Mixed model

$$L = \int d\bar{\theta}d\theta \left[ \frac{i}{2}(\bar{\mathcal{D}}\Phi) \frac{d}{dt}(\mathcal{D}\Phi) + \frac{\gamma}{2}\bar{\mathcal{D}}\Phi\mathcal{D}\Phi + V(\Phi) \right] .$$

Physics is similar to the model with  $\gamma = 0$ , but

- Not integrable anymore.
- No linear growth for  $D(t)$ .

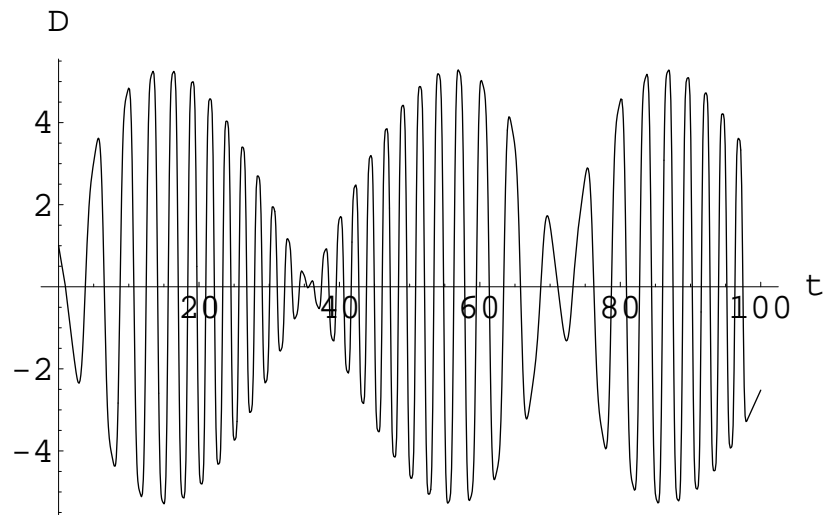


Figure 1: The function  $D(t)$  for a deformed system ( $\omega = 0, \lambda = 1, \gamma = .1$ ).



## UNUSUAL ALGEBRAIC STRUCTURES

- **canonical** Nöther supercharges

$$Q = \psi[p + iV'(x)] - \left(\bar{\chi} + \frac{\gamma}{2}\psi\right) (P - iD) ,$$

$$\bar{Q} = \left(\bar{\psi} - \frac{\gamma}{2}\chi\right) (P + iD) - \chi[p - iV'(x)] .$$

- and the **extra** pair

$$T = \psi[p - iV'(x)] + \left(\bar{\chi} + \frac{\gamma}{2}\psi\right) (P + iD) ,$$

$$\bar{T} = \left(\bar{\psi} + \frac{\gamma}{2}\chi\right) (P - iD) + \chi[p + iV'(x)] .$$

- When  $\gamma = 0$ , we have a semidirect product of the **standard**  $\mathcal{N} = 4$  SUSY algebra

$$\{Q, \bar{Q}\} = \{T, \bar{T}\} = 2H$$

(**but**  $\bar{Q} \neq Q^\dagger$ ,  $\bar{T} \neq T^\dagger$  !)

and the Abelian Lie algebra generated by

$$\begin{aligned} N &= \frac{P^2}{2} - V(\phi), \\ F &= \psi\bar{\psi} - \chi\bar{\chi} \end{aligned}$$

**Nonvanishing commutators**

$$\begin{aligned} \{Q, \bar{Q}\} &= \{T, \bar{T}\} = 2H; \\ [\bar{Q}, F] &= \bar{Q}, \quad [Q, F] = -Q, \quad [T, F] = -T, \quad [\bar{T}, F] = \bar{T}; \\ [Q, N] &= [T, N] = \frac{Q - T}{2}, \quad -[\bar{Q}, N] = [\bar{T}, N] = \frac{\bar{Q} + \bar{T}}{2}. \end{aligned}$$

- When  $\gamma \neq 0$ , the algebra is **deformed**:
- Let  $H = H_0 - \gamma F/2$  and introduce  $F_+ = \bar{\chi}\psi$ ,  $F_- = \bar{\psi}\chi$   
then

$$\begin{aligned}
[F_{\pm}, F] &= \mp 2F_{\pm}, & [F_+, F_-] &= F, \\
[Q, H_0] &= -\frac{\gamma}{2}Q, & [\bar{Q}, H_0] &= \frac{\gamma}{2}\bar{Q}, \\
[T, H_0] &= \frac{\gamma}{2}T, & [\bar{T}, H_0] &= -\frac{\gamma}{2}\bar{T}, \\
[Q, F] &= -Q, & [\bar{Q}, F] &= \bar{Q}, \\
[T, F] &= T, & [\bar{T}, F] &= -\bar{T}, \\
[Q, F_-] &= \bar{T}, & [\bar{Q}, F_+] &= -T, \\
[T, F_-] &= -\bar{Q}, & [\bar{T}, F_+] &= Q, \\
\{Q, \bar{Q}\} &= 2H_0 - \gamma F, & \{T, \bar{T}\} &= 2H_0 + \gamma F, \\
\{Q, T\} &= 2\gamma F_+, & \{\bar{Q}, \bar{T}\} &= 2\gamma F_-.
\end{aligned}$$

- This is  $osp(2, 2)$  algebra.
- a close relative of [weak supersymmetry](#) algebra [A.S., PLB 585 (2004) 173].

## (1+1) FIELD THEORY

- Let  $\Phi$  depend on  $t$  and  $x$ . Choose

$$S = \int dt dx d\bar{\theta} d\theta [-2i\mathcal{D}\Phi\partial_+\mathcal{D}\Phi + V(\Phi)] ,$$

where  $\partial_{\pm} = (\partial_t \pm \partial_x)/2$  and

$$\mathcal{D} = \frac{\partial}{\partial\theta} + i\theta\partial_-, \quad \bar{\mathcal{D}} = \frac{\partial}{\partial\bar{\theta}} - i\bar{\theta}\partial_+$$

### Bosonic Lagrangian

$$\mathcal{L}_B = \partial_{\mu}\phi\partial_{\mu}D + DV'(\phi)$$

with

$$V(\phi) = \frac{\omega^2\phi^2}{2} + \frac{\lambda\phi^4}{4}, \quad \lambda > 0.$$

Equations of motion:

$$\begin{aligned} \square\phi + \omega^2\phi + \lambda\phi^3 &= 0 \\ \square D + D(\omega^2 + 3\lambda\phi^2) &= 0. \end{aligned} \quad (4)$$

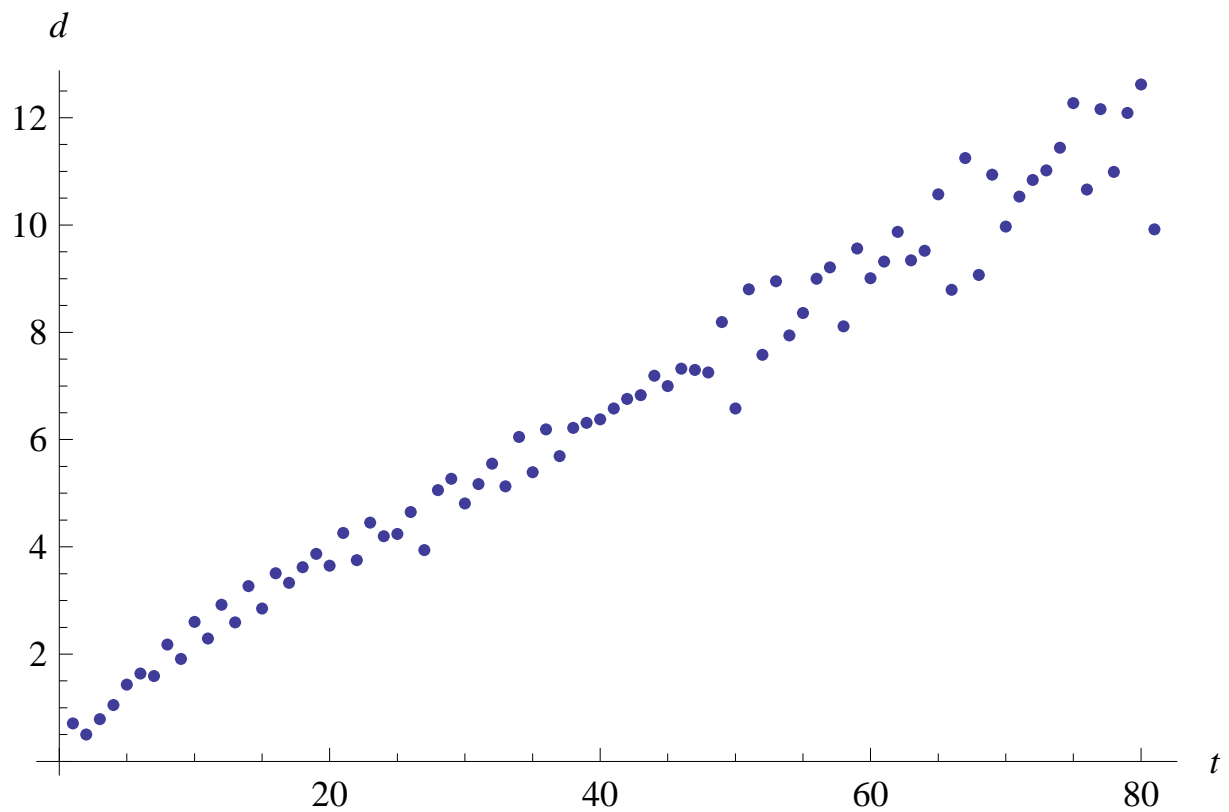
- Only two integrals of motion: the energy and

$$N = \int dx \left\{ \frac{1}{2} \left[ \dot{\phi}^2 + (\partial_x\phi)^2 \right] + \frac{\omega^2\phi^2}{2} + \frac{\lambda\phi^4}{4} \right\} .$$

- Not exactly solvable.

- **Stochasticity**. Solved numerically.
- Finite spatial box. Different initial conditions.

### TYPICAL BEHAVIOR:



**Dispersion**  $d = \sqrt{\langle D^2 \rangle_x}$  as a function of time.

## SWEET DREAM

*The TOE is a higher-derivative field theory with benign ghosts living in a higher-dimensional flat space-time. Our Universe represents a 3-brane — a solitonic solution extended in three spatial and the time directions and localized in the extra dimensions. Gravity arises as effective theory on the world volume of this brane.*