Weak pseudo-bosons

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Organization of the talk

1. *A (very) short review on $\mathcal{D}$-pseudo bosons*

2. *Weak pseudo bosons*

3. *Applications:*

   - position and momentum operators

   - the damped harmonic oscillator

$\mathcal{D}$-pseudo bosons

Let a and b be two operators on $\mathcal{H}$, $a^\dagger$ and $b^\dagger$ their adjoint, and let $\mathcal{D}$, dense in $\mathcal{H}$, be such that $a^\sharp\mathcal{D} \subseteq \mathcal{D}$ and $b^\sharp\mathcal{D} \subseteq \mathcal{D}$, $(x^\sharp = x, x^\dagger)$. In general $\mathcal{D} \subseteq D(a^\sharp)$ and $\mathcal{D} \subseteq D(b^\sharp)$.

Definition 1:

The operators (a, b) are $\mathcal{D}$-pseudo bosonic ($\mathcal{D}$-pb) if, for all $f \in \mathcal{D}$, we have

$$a\,b\,f - b\,a\,f = f. \tag{1}$$

($[a, b] = 1\!\!1$, for simplicity). [If $b = a^\dagger$ then we recover the CCR].

We now assume that

Assumption $\mathcal{D}$-pb 1.– there exists a non-zero $\varphi_0 \in \mathcal{D}$ such that $a\varphi_0 = 0$,

Assumption $\mathcal{D}$-pb 2.– there exists a non-zero $\Psi_0 \in \mathcal{D}$ such that $b^\dagger\Psi_0 = 0$.

Remark:– If $a = \frac{d}{dx}$ and $b = x$ these assumptions are not satisfied in $\mathcal{L}^2(\mathbb{R})$.

Now, if (a, b) satisfy Definition 1, then $\varphi_0 \in D^\infty(b)$ and $\Psi_0 \in D^\infty(a^\dagger)$. Hence...

$$\varphi_n := \frac{1}{\sqrt{n!}}\, b^n \varphi_0, \qquad \Psi_n := \frac{1}{\sqrt{n!}}\, {a^\dagger}^n \Psi_0, \tag{2}$$

$n \geq 0$, can be defined and they all belong to $\mathcal{D}$. We introduce, as before, $\mathcal{F}_\Psi = \{\Psi_n,\, n \geq 0\}$ and $\mathcal{F}_\varphi = \{\varphi_n,\, n \geq 0\}$. Once again, since $\mathcal{D}$ is stable under the action of $a^\sharp$ and $b^\sharp$, we deduce that both φ_n and Ψ_n belong to the domains of $a^\sharp$, $b^\sharp$ and $N^\sharp$ (here $N = ba$).

The following lowering and raising relations hold:

$$\left\{\begin{array}{ll} b\,\varphi_n = \sqrt{n+1}\varphi_{n+1}, & n \geq 0, \\ a\,\varphi_0 = 0, \quad a\varphi_n = \sqrt{n}\,\varphi_{n-1}, & n \geq 1, \\ a^\dagger \Psi_n = \sqrt{n+1}\Psi_{n+1}, & n \geq 0, \\ b^\dagger \Psi_0 = 0, \quad b^\dagger \Psi_n = \sqrt{n}\,\Psi_{n-1}, & n \geq 1, \end{array}\right. \tag{3}$$

as well as the following eigenvalue equations:

$$N\varphi_n = n\varphi_n, \quad N^\dagger \Psi_n = n\Psi_n, \quad n \geq 0.$$

A consequence: if $\langle \varphi_0, \Psi_0 \rangle = 1$, then

$$\langle \varphi_n, \Psi_m \rangle = \delta_{n,m}, \tag{4}$$

for all $n, m \geq 0$.

Assumption $\mathcal{D}$-pb 3.– $\mathcal{F}_\varphi$ is a basis for $\mathcal{H}$. (iff $\mathcal{F}_\Psi$ is a basis for $\mathcal{H}$)

Remarks:– (1) But, it is not o.n., in general, and for non o.n. sets completeness$\neq$basis!!

(2) If $b = a^\dagger$ (i.e. for CCR) all assumptions are satisfied. Also, $\mathcal{F}_\varphi = \mathcal{F}_\Psi$.

Sometimes (i.e. in concrete physical models) it is more convenient to check the following

Assumption $\mathcal{D}$-pbw 3.– $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are $\mathcal{D}$-quasi bases for $\mathcal{H}$.

This means that, $\forall\, f, g \in \mathcal{D}$,

$$\sum \langle f, \varphi_n\rangle \langle \Psi_n, g\rangle = \sum \langle f, \Psi_n\rangle \langle \varphi_n, g\rangle = \langle f, g\rangle\,.$$

Notice that... if $\{e_n\}$ is an o.n. basis for $\mathcal{H}$, and if S is a bounded operator, invertible with bounded inverse S^{-1}, then the sets $\{Se_n\}$ and $\{(S^{-1})^\dagger e_n\}$ are biorthogonal Riesz bases...

... but, if S or S^{-1} are **not** bounded, the sets $\{Se_n\}$ and $\{(S^{-1})^\dagger e_n\}$ can still be $\mathcal{D}$-quasi bases. This is exactly what happens in many explicit models!

Applications:–

1. "Extended" harmonic oscillator(s) [1-d, 2-d, or more]

$$H = \frac{\nu}{2}\left(p_1^2 + x_1^2 + p_2^2 + x_2^2\right) + i\sqrt{2}\,(p_1 + p_2),$$

$$H = \frac{1}{2}(p_1^2 + x_1^2) + \frac{1}{2}(p_2^2 + x_2^2) + i\left[A(x_1 + x_2) + B(p_1 + p_2)\right],$$

and so on

2. Swanson model

$$H_\theta = \frac{1}{2}\left(p^2 + x^2\right) - \frac{i}{2}\,\tan(2\theta)\left(p^2 - x^2\right),$$

3. Non commutative 2-d systems

$$H = \frac{1}{2}(p_1^2 + x_1^2) + \frac{1}{2}(p_2^2 + x_2^2) + i\left[A(x_1 + x_2) + B(p_1 + p_2)\right],$$

with $[x_1, x_2] \neq 0$

More applications:–

4. Black-Scholes equation

From

$$\frac{\partial C}{\partial t} = -\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rS\frac{\partial C}{\partial S} - rC,$$

one deduces

$$H_{BS}\Psi(x) = \varepsilon\Psi(x), \qquad \text{where} \qquad H_{BS} = -\frac{1}{2}\sigma^2\frac{d^2}{dx^2} + \left(\frac{\sigma^2}{2} - r\right)\frac{d}{dx} + r\,1\!\!1.$$

Then $H = H_{BS} + V(x)$, where $V(x)$ is some *financially motivated* potential.

5. Deformed Jaynes-Cummings Model

6. Generalized Bogoliubov transformations

7. Deformed graphene

.... and more

Back to $(\hat{x}, \hat{D} = \frac{d}{dx})$

Let $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$ and

$$\hat{x}f(x) = xf(x), \quad (\hat{D}g)(x) = g'(x),$$

for all

$$f(x) \in D(\hat{x}) = \{h(x) \in \mathcal{L}^2(\mathbb{R}) : xh(x) \in \mathcal{L}^2(\mathbb{R})\},$$

and

$$g(x) \in D(\hat{D}) = \{h(x) \in \mathcal{L}^2(\mathbb{R} : h'(x) \in \mathcal{L}^2(\mathbb{R})\}.$$

Of course, the set of test functions $\mathcal{S}(\mathbb{R})$ satisfies $\mathcal{S}(\mathbb{R}) \subset D(\hat{x}) \cap D(\hat{D})$.
We have:

$$\hat{x}^\dagger = \hat{x}, \qquad \hat{D}^\dagger = -\hat{D},$$

and

$$[\hat{D}, \hat{x}]f(x) = f(x),$$

for all $f(x) \in \mathcal{S}(\mathbb{R})$. This suggests that $\hat{x}$ and $\hat{D}$ could be thought as $\mathcal{S}(\mathbb{R})$-pseudo bosons, since, in particular, $\mathcal{S}(\mathbb{R})$ is stable under their action, and the action of their adjoints.
However, the vacua of $a = \hat{D}$ and $b = \hat{x}$ are $\varphi_0(x) = 1$ and $\psi_0(x) = \delta(x)$, (with a suitable choice of the normalizations) which **do not belong** to $\mathcal{S}(\mathbb{R})$, or even to $\mathcal{L}^2(\mathbb{R})$.
Nevertheless...

Back to $(\hat{x}, \hat{D} = \frac{d}{dx})$

...first of all, b and $a^\dagger$ still act as raising operators:

$$\varphi_n(x) = \frac{b^n}{\sqrt{n!}}\,\varphi_0(x) = \frac{x^n}{\sqrt{n!}}, \qquad \psi_n(x) = \frac{(a^\dagger)^n}{\sqrt{n!}}\,\psi_0(x) = \frac{(-1)^n}{\sqrt{n!}}\,\delta^{(n)}(x),$$

for all $n = 0, 1, 2, 3, \ldots$. Here $\delta^{(n)}(x)$ is the n-th weak derivative of the Dirac delta function. Both $\varphi_n(x)$ and $\psi_n(x) \in \mathcal{S}'(\mathbb{R})$.
This suggests to consider $a^\dagger$ and b as linear operators acting on $\mathcal{S}'(\mathbb{R})$. This is possible since the action of $\hat{x}$ and $\hat{D}$ can be extended outside $\mathcal{L}^2(\mathbb{R})$, to $\mathcal{S}'(\mathbb{R})$.
Furthermore, $\mathcal{S}'(\mathbb{R})$ is stable under the action of these operators.
Hence the pseudo-bosonic commutation relation, originally defined as $[\hat{D}, \hat{x}]f(x) = [a, b]f(x) = f(x)$, for all $f(x) \in \mathcal{S}(\mathbb{R})$, can be naturally extended to to $\mathcal{S}'(\mathbb{R})$:

$$[a, b]\varphi(x) = \varphi(x),$$

for all $\varphi(x) \in \mathcal{S}'(\mathbb{R})$.
We find that

$$b\varphi_k(x) = \sqrt{k+1}\,\varphi_{k+1}(x), \qquad a^\dagger\psi_k(x) = \sqrt{k+1}\,\psi_{k+1}(x),$$

and

$$a\varphi_k(x) = \sqrt{k}\,\varphi_{k-1}(x), \qquad b^\dagger\psi_k(x) = \sqrt{k}\,\psi_{k-1}(x),$$

$k = 0, 1, 2, 3, \ldots$, with the understanding that $a\varphi_0(x) = b^\dagger\psi_0(x) = 0$.

Back to $(\hat{x}, \hat{D} = \frac{d}{dx})$

Then we find that, calling $N = ba$ and $N^\dagger = a^\dagger b^\dagger$,

$$N\varphi_k(x) = k\varphi_k(x), \qquad N^\dagger\psi_k(x) = k\psi_k(x),$$

which show that $\varphi_k(x)$ and $\psi_k(x)$ belong to the generalized domains of N and $N^\dagger$, respectively. These follow from the previous ladder relations, or directly:

$$N\varphi_k(x) = \hat{x}\,\hat{D}\,\frac{x^k}{\sqrt{k!}} = \hat{x}\,\frac{kx^{k-1}}{\sqrt{k!}} = \frac{kx^k}{\sqrt{k!}} = k\varphi_k(x).$$

Also:

$$N^\dagger\psi_k(x) = -\hat{D}\hat{x}\frac{(-1)^k}{\sqrt{k!}}\,\delta^{(k)}(x) = (-1)^{k+1}(x\delta^{(k)}(x))' = k\psi_k(x),$$

since $(x\delta^{(k)}(x))' = -k\delta^{(k)}(x)$ for all $k = 0, 1, 2, 3, \ldots$.

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since $(x\delta^{(k)}(x))' = -k\delta^{(k)}(x)$ for all $k = 0, 1, 2, 3, \ldots$.

Next step: biorthonormality of $\mathcal{F}_\varphi = \{\varphi_n(x)\}$ and $\mathcal{F}_\psi = \{\psi_n(x)\}$.....but in which sense? We are outside $\mathcal{L}^2(\mathbb{R})$! And not just for a *wrong* scalar product or, which is the same, a wrong metric operator.

Back to $(\hat{x}, \hat{D} = \frac{d}{dx})$

Biorthogonality means scalar product which, in turns, means Hilbert space. But we are working with $\mathcal{S}'(\mathbb{R})$, which is **not** an Hilbert space, at all! Nevertheless, for all $f(x), g(x) \in \mathcal{S}(\mathbb{R})$, we can write

$$\langle f, g \rangle = (\overline{f} * \tilde{g})(0),$$

where $\tilde{g}(x) = g(-x)$. Hence, scalar product $\equiv$ convolution.

We now define the scalar product between two elements $F(x), G(x) \in \mathcal{S}'(\mathbb{R})$ as the following convolution:

$$\langle F, G \rangle := (\overline{F} * \tilde{G})(0),$$

whenever this convolution exists...and we know this convolution does exist in many cases...
To compute $\langle F, G \rangle$, we must compute $(\overline{F} * \tilde{G})[f]$, $f(x) \in \mathcal{S}(\mathbb{R})$, and this can be computed by using the equality

$$(\overline{F} * \tilde{G})[f] = \langle F, G * f \rangle \,.$$

In our situation we have $F(x) = x^n$ and $G(x) = \delta^{(m)}(x)$, $n, m = 0, 1, 2, 3, \ldots$. Hence

$$(G * f)(x) = \int_{\mathbb{R}} \delta^{(m)}(y) f(x-y) dy = f^{(m)}(x),$$

where $f^{(m)}(x)$ is the ordinary m-th derivative of the test function $f(x)$.

Back to $(\hat{x}, \hat{D} = \frac{d}{dx})$

Then, recalling once more that $F(x) = x^n \simeq \varphi_n(x)$ and $G(x) = \delta^{(m)}(x) \simeq \psi_m(x)$,

$$(\overline{F} * \tilde{G})[f] = \langle F, G * f\rangle = \int_{\mathbb{R}} \overline{F(x)} f^{(m)}(x)\,dx = \int_{\mathbb{R}} x^n \frac{d^m f(x)}{dx^m}\,dx =$$
$$= (-1)^m \int_{\mathbb{R}} \frac{d^m x^n}{dx^m} f(x)\,dx.$$

Since

$$\frac{d^m x^n}{dx^m} = \begin{cases} 0 & \text{if } m > n \\ n! & \text{if } m = n \\ \frac{n!}{(n-m)!}\, x^{n-m} & \text{if } m < n, \end{cases}$$

we conclude that

$$(\overline{F} * \tilde{G})[f] = \begin{cases} 0 & \text{if } m > n \\ (-1)^n n! \int_{\mathbb{R}} f(x)\,dx & \text{if } m = n \\ (-1)^m \frac{n!}{(n-m)!} \int_{\mathbb{R}} x^{n-m} f(x)\,dx & \text{if } m < n, \end{cases}$$

from which we deduce that

$$(\overline{F} * \tilde{G})(x) = \begin{cases} 0 & \text{if } m > n \\ (-1)^n n! & \text{if } m = n \\ (-1)^m \frac{n!}{(n-m)!} x^{n-m} & \text{if } m < n, \end{cases}$$

and therefore that $(\overline{F} * \tilde{G})(0) = (-1)^n n! \delta_{n,m}$. Hence,

$$\langle \varphi_n, \psi_m \rangle = \delta_{n,m}, \tag{5}$$

as claimed before.

Back to $(\hat{x}, \hat{D} = \frac{d}{dx})$

And now: the basis property

Back to $(\hat{x}, \hat{D} = \frac{d}{dx})$

And now: the basis property

It is clear that it makes no sense to check if $\mathcal{F}_\varphi$ or $\mathcal{F}_\psi$, or both, are bases or $\mathcal{D}$-quasi bases. This is because none of the $\varphi_n(x)$ and $\psi_n(x)$ even belongs to $\mathcal{L}^2(\mathbb{R})$. However, the pair $(\mathcal{F}_\varphi, \mathcal{F}_\psi)$ can still be used to expand a certain class of functions, those which admit expansion in Taylor series. In fact we have

$$\sum_{n=0}^{\infty} \langle \psi_n, f \rangle \, \varphi_n(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left\langle \delta^{(n)}, f \right\rangle x^n = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\, x^n = f(x),$$

for all $f(x)$ admitting this kind of expansion. However, if we exchange the role of $\mathcal{F}_\psi$ and $\mathcal{F}_\varphi$, the result is more complicated:

$$\sum_{n=0}^{\infty} \langle \varphi_n, f \rangle \, \psi_n(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \langle x^n, f \rangle \, \delta^{(n)}(x).$$

This is, in principle, an infinite series of derivatives of delta sometimes called *dual Taylor series*. It is known that the series does not define in general an element of $D'(\mathbb{R})$, a distribution, (hence it cannot define a tempered distribution) except when the number of non zero moments of $f(x)$, $\langle x^n, f \rangle$, is finite. In this particular situation the series above returns a finite sum, which is indeed a tempered distribution.

Back to $(\hat{x}, \hat{D} = \frac{d}{dx})$

Next: intertwining relations

Back to $(\hat{x}, \hat{D} = \frac{d}{dx})$

Next: intertwining relations

First of all, we introduce the following subsets of $\mathcal{S}'(\mathbb{R})$:

$$D(S_\varphi) = \{F(x) \in \mathcal{S}'(\mathbb{R}) : (S_\varphi F)(x) \in \mathcal{S}'(\mathbb{R})\}$$

and

$$D(S_\psi) = \{F(x) \in \mathcal{S}'(\mathbb{R}) : (S_\psi F)(x) \in \mathcal{S}'(\mathbb{R})\}.$$

We call these sets the *generalized domains* of S_φ and S_ψ, respectively. We put

$$S_\varphi\left(\sum_{k=0}^{N} c_k \psi_k\right) = \sum_{k=0}^{N} c_k \varphi_k, \qquad S_\psi\left(\sum_{k=0}^{N} c_k \varphi_k\right) = \sum_{k=0}^{N} c_k \psi_k.$$

This implies that $\mathcal{L}_\varphi \subseteq D(S_\psi)$ and $\mathcal{L}_\psi \subseteq D(S_\varphi)$ and that $S_\varphi : \mathcal{L}_\psi \to \mathcal{L}_\varphi$, while $S_\psi : \mathcal{L}_\varphi \to \mathcal{L}_\psi$. In particular we have

$$S_\varphi S_\psi F = F, \qquad S_\psi S_\varphi G = G, \tag{6}$$

and

$$N S_\varphi G = S_\varphi N^\dagger G, \qquad N^\dagger S_\psi F = S_\psi N F, \tag{7}$$

for $F(x) \in \mathcal{L}_\varphi$, $G(x) \in \mathcal{L}_\psi$. Furthermore, it is possible to see that $D(S_\varphi) \supset \mathcal{L}_\psi$. For instance, if $F(x) = \chi_{[0,1]}(x) \notin \mathcal{L}_\psi$, the series $\sum_{n=0}^{\infty} \langle \varphi_n, F\rangle \varphi_n(x) = \sum_{n=0}^{\infty} \alpha_n x^n$, $\alpha_n = \frac{1}{n!}\langle x^n, F\rangle$, converges for all $x \in \mathbb{R}$. Hence $F \in D(S_\varphi)$.

The general settings

Let again a and b be two operators which, together with their adjoints $a^\dagger$ and $b^\dagger$, map a certain dense subset of $\mathcal{H}$, $\mathcal{D}$, into itself. Assume that a and b can be extended to larger set, $\mathcal{E} \supset \mathcal{H}$, which is again stable under their action, and under the action of their adjoints.

Definition:– The operators a and b are *weak* $\mathcal{E}$-pseudo bosonic if

$$[a, b]\, F = F, \tag{8}$$

for all $F \in \mathcal{E}$.

Assumption $\mathcal{E}$-wpb 1.– there exists a non-zero $\varphi_0 \in \mathcal{E}$ such that $a\, \varphi_0 = 0$.

Assumption $\mathcal{E}$-wpb 2.– there exists a non-zero $\Psi_0 \in \mathcal{E}$ such that $b^\dagger\, \Psi_0 = 0$.

The invariance of $\mathcal{E}$ under the action of the operators a, b, $a^\dagger$ and $b^\dagger$ implies that $\varphi_0 \in D^\infty(b) := \cap_{k\geq 0} D(b^k)$ and $\Psi_0 \in D^\infty(a^\dagger)$, in the sense of generalized domains, so that the vectors

$$\varphi_n := \frac{1}{\sqrt{n!}}\, b^n \varphi_0, \qquad \Psi_n := \frac{1}{\sqrt{n!}}\, {a^\dagger}^n \Psi_0, \tag{9}$$

$n \geq 0$, can be defined and they all belong to $\mathcal{E}$.

The general settings

Defining now the sets $\mathcal{F}_\psi = \{\psi_n,\, n \geq 0\}$ and $\mathcal{F}_\varphi = \{\varphi_n,\, n \geq 0\}$, from (8) and from the definition in (9) we easily deduce the same raising and lowering relations as for $\mathcal{D}$-pbs, together with the eigenvalue equations

$$N\varphi_n = n\varphi_n, \qquad N^\dagger \Psi_n = n\Psi_n,$$

$n \geq 0$, where, once more, $N^\dagger$ is identified with $a^\dagger b^\dagger$.

It is natural to assume now that

$$\langle \varphi_n, \Psi_m \rangle = \delta_{n,m}, \tag{10}$$

for all $n, m \geq 0$, i.e. that $\mathcal{F}_\Psi$ and $\mathcal{F}_\varphi$ are biorthonormal, with respect to a bilinear form $\langle ., . \rangle$ which extends the ordinary scalar product to $\mathcal{E}$.

Of course, it makes not much sense to require any strong version of the basis property for $\mathcal{F}_\Psi$ or $\mathcal{F}_\varphi$, in general. Hence we simply require is that a set $\mathcal{C} \subseteq \mathcal{E}$ exists, *sufficiently large*, such that

$$\langle F, G \rangle = \sum_{n=0}^{\infty} \langle F, \psi_n \rangle \langle \varphi_n, G \rangle = \sum_{n=0}^{\infty} \langle F, \varphi_n \rangle \langle \psi_n, G \rangle\,, \tag{11}$$

for all $F, G \in \mathcal{C}$.

The general settings

We further use $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ to introduce two operators, S_φ and S_ψ, as follows: let

$$D(S_\varphi) = \{F \in \mathcal{E} : S_\varphi F \in \mathcal{E}\}, \qquad D(S_\psi) = \{F(x) \in \mathcal{E} : S_\psi F \in \mathcal{E}\},$$

with

$$S_\varphi \left(\sum_{k=0}^{N} c_k \psi_k \right) = \sum_{k=0}^{N} c_k \varphi_k, \qquad S_\psi \left(\sum_{k=0}^{N} c_k \varphi_k \right) = \sum_{k=0}^{N} c_k \psi_k.$$

As before, $D(S_\varphi)$ and $D(S_\psi)$ are the generalized domains of S_φ and S_ψ, respectively. All the properties found when dealing with $\hat{x}$ and $\hat{D}$ are recovered:

$$\mathcal{L}_\varphi \subseteq D(S_\psi), \quad \mathcal{L}_\psi \subseteq D(S_\varphi), \quad S_\varphi : \mathcal{L}_\psi \to \mathcal{L}_\varphi, \quad S_\psi : \mathcal{L}_\varphi \to \mathcal{L}_\psi.$$

In particular we have

$$S_\varphi S_\psi F = F, \qquad S_\psi S_\varphi G = G,$$

and

$$N S_\varphi G = S_\varphi N^\dagger G, \qquad N^\dagger S_\psi F = S_\psi N F.$$

Hence S_φ is a *sort of* inverse of S_ψ, and they intertwine between N and $N^\dagger$ (which are of pair of number-like operators, similarly to what happens for $\mathcal{D}$-pbs).

Two words of motivation

1. In S. Deguchi, Y. Fujiwara, K. Nakano, *Two quantization approaches to the Bateman oscillator model*, Ann. of Phys., 2019, the authors propose a general approach based on ladder operators claiming they can find a basis in $\mathcal{L}^2(\mathbb{R}^2)$;
2. In F. Bagarello, F. Gargano, F. Roccati, *A no-go result for the quantum damped harmonic oscillator*, Phys. Lett. A, (2019) we proved that their result was wrong: their construction **cannot** work in $\mathcal{L}^2(\mathbb{R}^2)$, or in any other Hilbert space;
3. In S. Deguchi, Y. Fujiwara, *Square-integrable eigenfunctions in quantizing the Bateman oscillator model*, arXiv:1910.08271 they replied we were wrong and they were right. In fact, they claimed they were able to find explicitly the eigenfunctions of the Hamiltonian, and they are all square-integrable.
4. Their argument was based on the existence of a (**unbounded**) similarity operator mapping the Hamiltonian of a a two-dimensional harmonic oscillator to the Bateman Hamiltonian.
5. In F. Bagarello, F. Gargano, F. Roccati, *Some remarks on few recent results on the damped quantum harmonic oscillator*, Ann. of Phys., (2020) we proved they were wrong in two ways:
   1. their computations were simply wrong: e.g., $A_1\varphi_{0,0}(x_1,x_2)\neq 0$!
   2. their general argument, despite being rather natural, is also quite dangerous. This is typical when dealing with unbounded operators. In fact....

An interlude on $(\hat{x}, \hat{p})$ as unbounded operators

We want to show what can be wrong with unbounded operators in a very simple model connected to our operators.

Let $\hat{x}$ and $\hat{p}$ be the position and momentum operators, $[\hat{x}, \hat{p}] = i\mathbb{1}$. We put

$$c = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p}) \qquad \Rightarrow \qquad [c, c^\dagger] = \mathbb{1}.$$

The vacuum of c, $c\varphi_0(x) = 0$, satisfies the differential equation

$$\varphi_0'(x) = -x\varphi_0(x) \qquad \Rightarrow \qquad \varphi_0(x) = Ne^{-x^2/2} \in \mathcal{L}^2(\mathbb{R}).$$

It is also well known that no square-integrable vacuum exists for $c^\dagger = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p})$, since the solution of $c^\dagger\psi_0(x) = 0$ is proportional to $e^{x^2/2}$.

Let us consider the operator $T = e^{\frac{7\pi}{8}\left(c^2 + (c^\dagger)^2\right)}$. This operator is unbounded, invertible, and (formally) self-adjoint. We want to show that working with T as if it was a bounded operator creates paradoxes. Hence, from now on, we will work formally, paying no attention to domains of operators and we will deduce *something strange*.

An interlude on $(\hat{x}, \hat{p})$ as unbounded operators

First of all, it is easy to check that

$$\hat{x} = \frac{1}{\sqrt{2}}(c + c^\dagger) = TcT^{-1}.$$

Now, defining $\Phi_0(x) = T\varphi_0(x)$, we should have,

$$\hat{x}\,\Phi_0(x) = (TcT^{-1})(T\varphi_0(x)) = T\,c\,\varphi_0(x) = 0,$$

which admits as the only (strong) solution $\Phi_0(x) = 0$. But this is **not compatible** with the existence of T^{-1} and with the fact that $\varphi_0(x) \neq 0$, since

$$0 \neq \varphi_0(x) = T^{-1}\Phi_0(x) = 0.$$

Of course, a non trivial solution of $x\,\Phi_0(x) = 0$ does exist, but only in a distributional sense:

$$\Phi_0(x) = N'\delta(x).$$

An interlude on $(\hat{x}, \hat{p})$ as unbounded operators

The same conclusion can be deduced by noticing that, with a little algebra,

$$T\varphi_0 = e^{-\frac{1}{2}c^{\dagger 2}}\, e^{\frac{1}{4}\log 2(cc^\dagger + c^\dagger c)}\, e^{-\frac{1}{2}c^2}\varphi_0 =$$

$$= 2^{1/4} e^{-\frac{1}{2}c^{\dagger 2}}\varphi_0 = 2^{1/4}\sum_{k=0}^{\infty}\frac{\sqrt{2k!}}{k!}\left(-\frac{1}{2}\right)^k \varphi_{2k}.$$

Here $\{\varphi_k\}_{k\geq 0}$ is the basis of $\mathcal{L}^2(\mathbb{R})$ made by the eigenstates of the quantum harmonic oscillator. It is possible to check that the series for $\|T\varphi_0\|^2$ diverges. Hence $T\varphi_0$ is not a vector in $\mathcal{L}^2(\mathbb{R})$, in agreement with what we have explicitly deduced above.

Bateman lagrangian: classical system

The Bateman lagrangian is

$$L = m\dot{x}\dot{y} + \frac{\gamma}{2}(x\dot{y} - \dot{x}y) - kxy, \tag{12}$$

which returns the equations

$$m\ddot{x} + \gamma\dot{x} + kx = 0$$

and

$$m\ddot{y} - \gamma\dot{y} + ky = 0,$$

where m, γ and k are the physical positive quantities of the oscillator. The first equation is associated to the damped harmonic oscillator (DHO), while the second to a virtual amplified oscillator, which gain what is lost by the first. The conjugate momenta are

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{y} - \frac{\gamma}{2}\,y, \qquad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{x} + \frac{\gamma}{2}\,y,$$

and the corresponding classical Hamiltonian is

$$H = p_x\dot{x} + p_y\dot{y} - L = \frac{1}{m}p_xp_y + \frac{\gamma}{2m}\left(yp_y - xp_x\right) + \left(k - \frac{\gamma^2}{4m}\right)xy. \tag{13}$$

Bateman lagrangian: classical system

By introducing the new variables x_1 and x_2 through

$$x = \frac{1}{\sqrt{2}}(x_1 + x_2), \qquad y = \frac{1}{\sqrt{2}}(x_1 - x_2), \tag{14}$$

L and H can be written as follows:

$$L = \frac{m}{2}(\dot{x}_1^2 - \dot{x}_2^2) + \frac{\gamma}{2}(x_2\dot{x}_1 - x_1\dot{x}_2) - \frac{k}{2}(x_1^2 - x_2^2)$$

and

$$H = \frac{1}{2m}\left(p_1 - \frac{\gamma}{2}x_2\right)^2 - \frac{1}{2m}\left(p_2 - \frac{\gamma}{2}x_1\right)^2 + \frac{k}{2}(x_1^2 - x_2^2),$$

where $p_1 = \frac{\partial L}{\partial \dot{x}_1} = m\dot{x}_1 + \frac{\gamma}{2}x_2$ and $p_2 = \frac{\partial L}{\partial \dot{x}_2} = m\dot{x}_2 - \frac{\gamma}{2}x_1$. We introduce next $\omega^2 = \frac{k}{m} - \frac{\gamma^2}{4m^2}$, which we assume here to be strictly positive. We can rewrite H as follows:

$$H = \left(\frac{1}{2m}p_1^2 + \frac{1}{2}m\omega^2 x_1^2\right) - \left(\frac{1}{2m}p_2^2 + \frac{1}{2}m\omega^2 x_2^2\right) - \frac{\gamma}{2m}(p_1x_2 + p_2x_1). \tag{15}$$

Bateman lagrangian: canonical quantization

We now impose the following canonical quantization rules between x_j and p_k: $[x_j, p_k] = i\delta_{j,k}1\!\!1$. Ladder operators can now be easily introduced:

$$a_k = \sqrt{\frac{m\omega}{2}}\, x_k + i\sqrt{\frac{1}{2m\omega}}\, p_k, \tag{16}$$

$k = 1, 2$. These are bosonic operators since they satisfy the canonical commutation rules: $[a_j, a_k^\dagger] = \delta_{j,k}1\!\!1$. In terms of these operators the quantum version of the Hamiltonian H in (15) can be written as

$$H = \omega\left(a_1^\dagger a_1 - a_2^\dagger a_2\right) + \frac{i\gamma}{2m}\left(a_1 a_2 - a_1^\dagger a_2^\dagger\right) \tag{17}$$

We can rewrite H in a diagonal form by introducing

$$A_1 = \frac{1}{\sqrt{2}}(a_1 - a_2^\dagger), \quad A_2 = \frac{1}{\sqrt{2}}(-a_1^\dagger + a_2),$$

and

$$B_1 = \frac{1}{\sqrt{2}}(a_1^\dagger + a_2), \quad B_2 = \frac{1}{\sqrt{2}}(a_1 + a_2^\dagger).$$

Bateman lagrangian: canonical quantization

These operators satisfy the following requirements:

$$[A_j, B_k] = \delta_{j,k} \mathbb{1}, \tag{18}$$

with $B_j \neq A_j^\dagger$, $j = 1, 2$.
In fact, in terms of these operators, H can now be written as follows:

$$H = \omega \left(B_1 A_1 - B_2 A_2 \right) + \frac{i\gamma}{2m} \left(B_1 A_1 + B_2 A_2 + \mathbb{1} \right), \tag{19}$$

which only depends on the pseudo-bosonic number operators $N_j = B_j A_j$. The next proposition shows why wpbs are useful to deal with the DHO.

Connections with WPBs

Proposition:– There is no non-zero function $\varphi_{00}(x_1, x_2)$ satisfying

$$A_1\varphi_{00}(x_1, x_2) = A_2\varphi_{00}(x_1, x_2) = 0. \tag{20}$$

Also, there is no non-zero function $\psi_{00}(x_1, x_2)$ satisfying

$$B_1^\dagger\psi_{00}(x_1, x_2) = B_2^\dagger\psi_{00}(x_1, x_2) = 0. \tag{21}$$

The key of the proof is that the solution of (20) and (21) must be of the form

$$\varphi_{00}(x_1, x_2) = \alpha\delta(x_1 - x_2), \qquad \psi_{00}(x_1, x_2) = \beta\delta(x_1 + x_2), \qquad \alpha, \beta \in \mathbb{C}.$$

Comments:– (1) The result is independent on the choice of the metric in $\mathcal{L}^2$, also because changing the metric changes the expression of the adjoint!

(2) Already Feshback, in 1997, pointed out the impossibility of normalizing the eigenvectors in $\mathcal{L}^2$.

(3) An Hilbert space description is not so natural. It seems more convenient enlarge the functional settings to distributions (where wpbs are relevant) or, maybe, to rigged Hilbert spaces.

(4) What about other dissipative systems?.....Work in progress

Conclusions

Some references:

1. F. Bagarello, J. P. Gazeau, F. H. Szafraniec e M. Znojil Eds., *Non-selfadjoint operators in quantum physics: Mathematical aspects*, John Wiley and Sons (2015)
2. F. Bagarello, F. Gargano, F. Roccati, *A no-go result for the quantum damped harmonic oscillator*, Phys. Lett. A, **383**, 2836-2838 (2019)
3. F. Bagarello, F. Gargano, F. Roccati, *Some remarks on few recent results on the damped quantum harmonic oscillator*, Ann. of Phys., **414**, 168091, (2020)
4. F. Bagarello, *Weak pseudo-bosons*, J. Phys. A, **53**, 135201 (2020)

Conclusions

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1. F. Bagarello, J. P. Gazeau, F. H. Szafraniec e M. Znojil Eds., *Non-selfadjoint operators in quantum physics: Mathematical aspects*, John Wiley and Sons (2015)
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3. F. Bagarello, F. Gargano, F. Roccati, *Some remarks on few recent results on the damped quantum harmonic oscillator*, Ann. of Phys., **414**, 168091, (2020)
4. F. Bagarello, *Weak pseudo-bosons*, J. Phys. A, **53**, 135201 (2020)

....the end!!!