Evolution speed of open quantum dynamics

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A brief history of PTQM

- 1998 (Bender & Boettcher): PT-invariance $\Rightarrow$ Possibly real eigenvalues
- Issue: $P^2 = 1$ and $\mathrm{tr}(P) = 0$, so a Hilbert space with a PT-conjugation is indefinite
- 2002 (Bender *at al.*, Mostafazadeh): CPT-invariance $\Rightarrow$ Positive-definite inner product
- Question: Is PTQM $\equiv$ Hermitian QM?
- Answer not so straightforward (different degrees of freedom, existence of symmetry breaking, etc.)
- 2014 (Brody): In finite dimensions, PTQM $\equiv$ Hermitian QM
- In infinite dimensions, the situation is different
- $\sim$2007: Realisation of PT symmetry in a lab $\Rightarrow$ open system dynamics

Open quantum systems

An open system S is one that is coupled to another system B.

The total system $S+B$ may be described by a wave function $|\Psi_{S+B}\rangle$.

The state of an open quantum system S is then determined by tracing out B:

$$\hat{\rho}_S = \mathrm{tr}_B \left(\frac{|\Psi_{S+B}\rangle\langle\Psi_{S+B}|}{\langle\Psi_{S+B}|\Psi_{S+B}\rangle} \right)$$

The result is a mixed-state density matrix: $\hat{\rho}_S^2 \neq \hat{\rho}_S$.

For example, if S and B are both spin-$\frac{1}{2}$ particles, and $|\Psi_{S+B}\rangle$ is the spin-0 singlet state, then

$$\hat{\rho}_S = \tfrac{1}{2}\big(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\big) \quad \neq \quad \tfrac{1}{\sqrt{2}}\big(|\uparrow\rangle + |\downarrow\rangle\big)\,!$$

We can think of a density matrix as the (classical) statistical average over pure states:

$$\hat{\rho} = \left\langle \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle} \right\rangle$$

Dynamics of open quantum systems

The time evolution of the density matrix that preserves its positivity and trace condition is necessarily of the Lindblad form:

$$\partial_t \hat{\rho} = -\mathrm{i}[\hat{H}, \hat{\rho}] + \sum_k \left[\hat{L}_k \hat{\rho} \hat{L}_k^\dagger - \frac{1}{2} \left(\hat{L}_k^\dagger \hat{L}_k \rho + \hat{\rho} \hat{L}_k^\dagger \hat{L}_k \right) \right]$$

This is the so-called GKLS equation of Gorini, Kossakowski, Sudarshan (1976) and Lindblad (1976), or just the Lindblad equation for short.

From the linearity of the dynamics we can think of the right side as representing the action of a Liouville operator $\mathcal{L}$ on $\hat{\rho}$, and write

$$\partial_t \hat{\rho} = \mathcal{L} \hat{\rho}$$

It turns out that the eigenvalues of $\mathcal{L}$ are real or else come in complex conjugate pairs.

$\Rightarrow$ Every open system dynamics is, in a certain cense, PT symmetric.

Evolution speed (pure states)

The speed of the dynamics of a pure state is determined by use of the Schrödinger-Kibble equation

$$\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = -\mathrm{i}\frac{1}{\hbar}\left(\hat{H} - \langle\psi(t)|\hat{H}|\psi(t)\rangle\right)|\psi(t)\rangle$$

Then the squared speed of evolution is then determined by

$$v^2(t) = 4\langle\dot{\psi}(t)|\dot{\psi}(t)\rangle$$

where the factor of 4 is conventional.

Alternatively, this can be seen if we define the proper 'velocity' vector by

$$|v(t)\rangle = |\dot{\psi}(t)\rangle - \langle\psi(t)|\dot{\psi}(t)\rangle\,|\psi(t)\rangle$$

with $|\dot{\psi}(t)\rangle = -\mathrm{i}\hbar^{-1}\hat{H}|\psi(t)\rangle$, and calculate $v^2(t) = 4\langle v(t)|v(t)\rangle$.

Either way, we deduce the Anandan-Aharanov relation:

$$v^2(t) = \frac{4\Delta H^2}{\hbar^2}$$

which is independent of t.

Mixed state evolution speed (unitary case)

In the case of a mixed state, the speed of evolution is somewhat slowed down.

To work out the evolution speed we consider the Hermitian square-root of the density matrix:

$$\hat{\rho} \to \hat{\xi} = \sqrt{\hat{\rho}}$$

For a one-parameter family of unitary orbit $\hat{\xi}(t)$, the speed of evolution is determined by

$$v^2(t) = 4\,\mathrm{tr}(\partial_t\hat{\xi}\,\partial_t\hat{\xi})$$

A calculation shows that

$$\begin{aligned} \mathrm{tr}(\partial_t\hat{\xi}\partial_t\hat{\xi}) &= 2\left[\mathrm{tr}(\hat{H}^2\hat{\rho}) - \mathrm{tr}(\hat{H}\sqrt{\hat{\rho}}\hat{H}\sqrt{\hat{\rho}})\right] \\ &= \mathrm{tr}\left[\sqrt{\hat{\rho}}, \hat{H}\right]^2, \end{aligned}$$

which is twice the Wigner-Yanase skew information.

Embedding mixed states in Euclidean space

The space of density matrices in a Hilbert space $\mathcal{H}^n$ of dimension n forms a subset of the interior of a sphere S^{n^2-2} in a Euclidean space $\mathbb{R}^{n^2-1}$.

Thus every density matrix $\hat{\rho}$ can be thought of as being represented by a vector $\boldsymbol{r} \in \mathbb{R}^{n^2-1}$.

We let $\{\hat{\sigma}_j\}_{j=0,\ldots,n^2-1}$ be an orthonormal basis for the linear space of bounded operators on $\mathcal{H}^n$ with the Hilbert-Schmidt inner product $\langle \hat{\sigma}, \hat{\tau} \rangle = \mathrm{tr}\left(\hat{\sigma}^\dagger \hat{\tau}\right)$.

We set $\hat{\sigma}_0 = n^{-1/2}\mathbb{1}$, hence the operators $\{\hat{\sigma}_j\}_{j=1,\ldots,n^2-1}$ are trace free, and together they satisfy the orthonormality condition

$$\langle \hat{\sigma}_i, \hat{\sigma}_j \rangle = \delta_{ij}.$$

An arbitrary density matrix $\hat{\rho}$ can then be expressed in the form

$$\hat{\rho} = \frac{1}{\sqrt{n}}\hat{\sigma}_0 + \sum_{j=1}^{n^2-1} r_j\,\hat{\sigma}_j,$$

where $r_j = \mathrm{tr}(\hat{\rho}\hat{\sigma}_j)$.

For a pure state we have

$$1 = \mathrm{tr}(\hat{\rho}^2) = \frac{1}{n} + \sum_{j=1}^{n^2-1} r_j^2,$$

from which it follows that the squared radius of the sphere S^{n^2-2} in $\mathbb{R}^{n^2-1}$ is given by $1 - n^{-1}$.

Consider a one-parameter family of density matrices $\hat{\rho}(t)$ satisfying

$$\partial_t \hat{\rho} = -\mathrm{i}[\hat{H}, \hat{\rho}] + \sum_k \left[\hat{L}_k \hat{\rho} \hat{L}_k^\dagger - \frac{1}{2} \left(\hat{L}_k^\dagger \hat{L}_k \rho + \hat{\rho} \hat{L}_k^\dagger \hat{L}_k \right) \right]$$

along with an initial condition $\hat{\rho}(0)$.

Substituting

$$\hat{\rho} = \frac{1}{\sqrt{n}} \hat{\sigma}_0 + \sum_{j=1}^{n^2-1} r_j \, \hat{\sigma}_j$$

in here, we find

$$\dot{r}_i = \sum_{j=1}^{n^2-1} \Lambda_{ij} r_j + b_i,$$

where

$$\Lambda_{ij} = \operatorname{tr}\left[-\mathrm{i}\,[\hat{\sigma}_j, \hat{\sigma}_i]\hat{H} + \sum_k \hat{L}_k \hat{\sigma}_j \hat{L}_k^\dagger \hat{\sigma}_i - \tfrac{1}{2}\sum_k \left(\hat{L}_k^\dagger \hat{L}_k \hat{\sigma}_j \hat{\sigma}_i + \hat{L}_k^\dagger \hat{L}_k \hat{\sigma}_i \hat{\sigma}_j\right)\right]$$

is a real matrix, and

$$b_i = \frac{1}{n}\sum_k \operatorname{tr}\left([\hat{L}_k, \hat{L}_k^\dagger]\hat{\sigma}_i\right)$$

is a real vector.

Writing the dynamical equation in the form $\partial_t \hat{\rho} = \mathcal{L}\hat{\rho}$, the components $\mathcal{L}_{ij}$ of $\mathcal{L}$ in the basis $\{\hat{\sigma}_j\}$ are

$$\mathcal{L}_{ij} = \langle \hat{\sigma}_i, \mathcal{L}\hat{\sigma}_j \rangle = \operatorname{tr}\left(\hat{\sigma}_i \mathcal{L}\hat{\sigma}_j\right),$$

and we have $(\mathcal{L}\hat{\rho})_j = \sum_i \operatorname{tr}(\hat{\sigma}_j \mathcal{L}\hat{\sigma}_i)\operatorname{tr}(\hat{\sigma}_i\hat{\rho})$.

Therefore, writing

$$\dot{r}_j = \sum_{i=1}^{n^2-1} \mathcal{L}_{ji} r_i + \frac{1}{\sqrt{n}} \mathcal{L}_{j0}$$

we deduce that $\mathcal{L}_{ji} = \Lambda_{ji}$ for $i, j \neq 0$ and $\mathcal{L}_{j0} = \sqrt{n}\, b_j$.

Because the real matrix $\mathcal{L}_{ji}$ is not symmetric, its eigenvalues are either real or else come in complex conjugate pairs.

Evolution speed for mixed states

The squared speed of evolution of the state in $\boldsymbol{r} \in \mathbb{R}^{n^2-1}$ is

$$v^2(t) = \sum_{j=1}^{n^2-1} \dot{r}_j^2 = \mathrm{tr}\left[(\mathcal{L}\hat{\rho})^2\right].$$

Writing

$$\mathcal{L}\hat{\rho} = -\mathrm{i}[\hat{H},\hat{\rho}] + \mathcal{D}\hat{\rho},$$

where

$$\mathcal{D}\hat{\rho} = \sum_k \left[\hat{L}_k\hat{\rho}\hat{L}_k^\dagger - \frac{1}{2}\left(\hat{L}_k^\dagger\hat{L}_k\rho + \hat{\rho}\hat{L}_k^\dagger\hat{L}_k\right)\right],$$

we obtain:

$$v^2(t) = 2\left[\mathrm{tr}\left(\hat{H}^2\hat{\rho}^2\right) - \mathrm{tr}\left(\hat{H}\hat{\rho}\hat{H}\hat{\rho}\right)\right] - 2\mathrm{i}\,\mathrm{tr}\left(\hat{\rho}\,[\mathcal{D}\hat{\rho},\hat{H}]\right) + \mathrm{tr}\left[(\mathcal{D}\hat{\rho})^2\right].$$

There are three terms contributing to the speed of evolution.

The contribution to v^2 from the unitary evolution resembles, but is different from, the Wigner-Yanase skew information $I = \mathrm{tr}(\hat{H}^2\hat{\rho}) - \mathrm{tr}(\hat{H}\sqrt{\hat{\rho}}\hat{H}\sqrt{\hat{\rho}})$.

We call $S(X) = \mathrm{tr}(\hat{X}^\dagger \hat{X} \hat{\rho}^2) - \mathrm{tr}(\hat{X} \hat{\rho} \hat{X}^\dagger \hat{\rho})$ the 'modified skew information' for $\hat{X}$, which reduces to $\Delta X^2 = \langle \hat{X}^\dagger \hat{X} \rangle - \langle \hat{X}^\dagger \rangle \langle \hat{X} \rangle$ for a pure state.

In contrast to unitary time evolution, in an open system the velocity will in general obtain a radial component so that the purity $\mathrm{tr}(\hat{\rho}^2)$ changes.

To see this, consider the squared magnitude of the radial velocity

$$v_R^2(t) = \frac{(\boldsymbol{r} \cdot \dot{\boldsymbol{r}})^2}{\boldsymbol{r} \cdot \boldsymbol{r}} = \frac{[\mathrm{tr}(\hat{\rho} \boldsymbol{\mathcal{L}} \hat{\rho})]^2}{\mathrm{tr}[(\hat{\rho} - n^{-1} \mathbb{1})^2]},$$

which vanishes for unitary dynamics.

A short calculation shows remarkably that

$$v_R(t) = \sum_k \frac{S(L_k)}{\sqrt{\mathrm{tr}\left[(\hat{\rho} - n^{-1} \mathbb{1})^2\right]}}.$$

In other words, the speed of the change of the purity is given by twice the modified skew information associated with the Lindblad operators.

Examples

We now examine the behaviour of the evolution speed via illustrative examples.

(i) For the first example we take the Hamiltonian to be $\hat{H} = \frac{1}{2} g \hat{\sigma}_z$ and the Lindblad operator to be $\hat{L} = \sqrt{\gamma}\hat{\sigma}_z$.

In this example we find that

$$v^2(t) = \mathrm{e}^{-4\gamma t}(4\gamma^2 + g^2)[r_x^2(0) + r_y^2(0)],$$

and hence that $v(t)$ decreases exponentially in time.

(ii) Suppose that $\hat{H} = \frac{1}{2} g \hat{\sigma}_x$ and $\hat{L} = \sqrt{\gamma}\hat{\sigma}_z$.

This is the simplest example of a PT-symmetric quantum system.

In this case we have

$$\Lambda = \begin{pmatrix} -2\gamma & 0 & 0 \\ 0 & -2\gamma & -g \\ 0 & g & 0 \end{pmatrix},$$

and the four eigenvalues of the Liouville operator are thus given by 0, -2γ, and

$-\gamma \pm \sqrt{\gamma^2 - g^2}$.

In the region of unbroken PT-symmetry $g > \gamma$ the eigenvalues are either real or come in complex conjugate pairs; at the exceptional point $g = \gamma$ the PT-symmetry gets broken; and in the broken phase $g < \gamma$ all the eigenvalues are real.

We expect to observe different behaviours of the system in each of these phases.

Indeed, the solutions to the dynamical equation are:

$$
\begin{aligned}
r_x(t) &= \mathrm{e}^{-2\gamma t} r_x(0), \\
r_y(t) &= \mathrm{e}^{-\gamma t}[(\cos\omega t - (\gamma/\omega)\sin\omega t)\, r_y(0) - (g/\omega)\sin\omega t\, r_z(0)], \\
r_z(t) &= \mathrm{e}^{-\gamma t}[(g/\omega)\sin\omega t\, r_y(0) + (\cos\omega t + (\gamma/\omega)\sin\omega t)\, r_z(0)],
\end{aligned}
$$

with $\omega = \sqrt{g^2 - \gamma^2}$.

They are oscillatory in the unbroken phase $g > \gamma$, whereas in the broken phase $g < \gamma$ the oscillations are completely suppressed.

The speed $v^2(t)$, $v_R^2(t)$, and $v_T^2(t) = v^2(t) - v_R^2(t)$, are shown below for a system prepared in the spin-z up state $|\psi(0)\rangle = |\uparrow\rangle$.

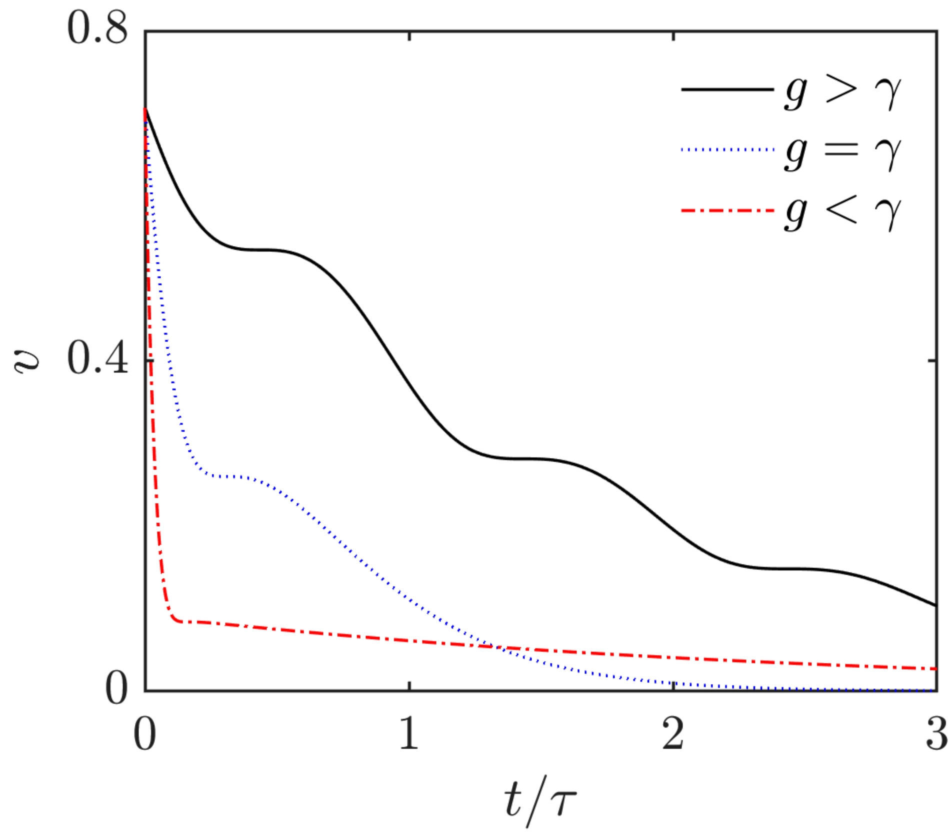
Because this is an eigenstate of $\hat{L}$, we have $v_R(t) = 0$ at $t = 0$.

In the unbroken phase, the speed exhibits a decay superimposed with oscillations.

Here $v_R(t)$ oscillates periodically with the period $\tau = \pi/\sqrt{g^2 - \gamma^2}$, where the minima correspond to the times at which the Bloch vector is aligned with the z-axis, i.e. when $\mathcal{D}\hat{\rho} \propto \hat{\rho}$.

Moving into the broken phase, the speed decays rapidly at short times and the oscillation in $v_T(t)$ is completely damped out.

However, in this phase the velocity remains nonzero for a longer duration.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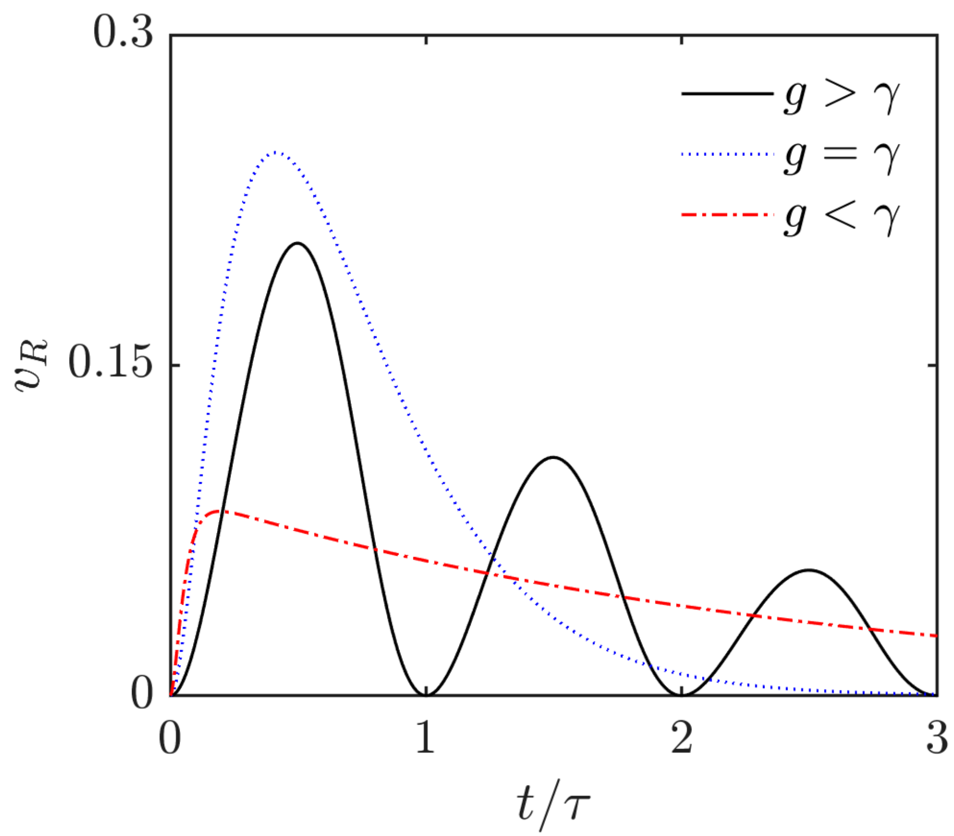
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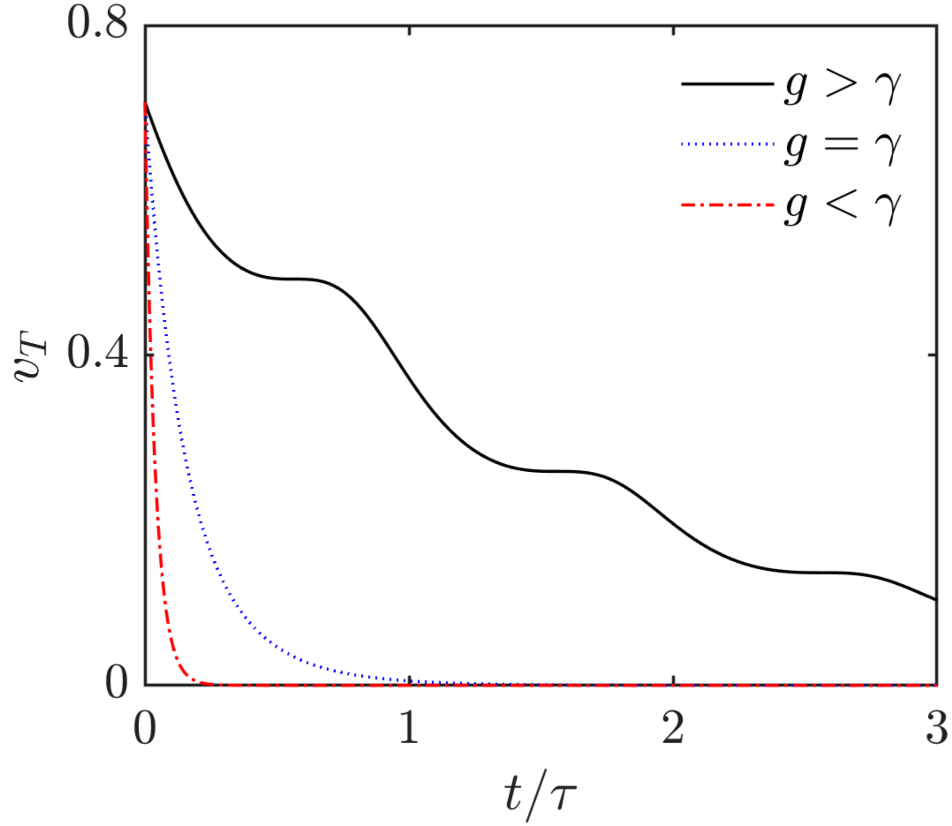
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